SCATTERING ON STRATIFIED MEDIA: THE MICRO-LOCAL PROPERTIES OF THE SCATTERING MATRIX AND RECOVERING ASYMPTOTICS OF PERTURBATIONS

T. CHRISTIANSEN AND M. S. JOSHI

Abstract. The fixed energy scattering matrix is defined on a perturbed stratified medium, and for a class of perturbations, its main part is shown to be a Fourier integral operator on the sphere at infinity. This is facilitated by developing a refined limiting absorption principle. The symbol of the scattering matrix is shown to determine the asymptotics of a large class of perturbations.

1. Introduction

In this paper, we study the structure of the scattering matrix on a perturbed stratified medium. In particular, we show that its main part is a Fourier integral operator. En route to proving this theorem, we develop an improved limiting absorption principle for a large class of perturbations, using techniques of Fourier and microlocal analysis. As an application of our results, we prove that the asymptotics of a perturbation can be recovered from the scattering matrix at one energy.

We recall that a stratified medium is a model space in which sounds waves propagate with a variable sound speed which depends on only one coordinate. Thus, if we write the coordinates on $\mathbb{R}^n$ as $z = (x, y)$ with $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$, we take the wave speed to be of the form $c_0(y)$ and study the wave equation

$$(-\frac{\partial^2}{\partial t^2} - c_0^2\Delta)w = 0, \tag{1}$$

where $\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial z_j^2}.$ We assume that c_0 is constant for $|y|$ large and that it is piecewise smooth. We let

$$c_+ = \lim_{y\to\infty} c_0(y), \tag{2}$$

$$c_- = \lim_{y\to-\infty} c_0(y). \tag{3}$$

In general, we do not require that c_+ be equal to c_-, but some of our results are stronger when they are equal.

A perturbed stratified medium is a medium on which the variable sound speed, c, has the property that $c-c_0$ is well-behaved at infinity. Many previous papers have studied the case where the perturbation $c-c_0$ is rapidly decaying. In particular, precise asymptotics for $(c^2\Delta-(\lambda-i0)^2)^{-1}f$, when $f \in \mathcal{S}(\mathbb{R}^n)$, were proved in [5]. The inverse scattering problem for exponentially decaying perturbations was studied in [1, 2, 14, 16, 29], where it was shown that under certain conditions, the perturbation can be recovered.

Here we study the case where the perturbation $c - c_0$ has an asymptotic expansion in homogeneous terms at infinity. Under certain conditions on c and c_0 made more precise in Section 2, we show that the scattering matrix for $c^2\Delta$ is a Fourier integral operator and describe its singular set. Moreover, we show that the asymptotics of the perturbation can be recovered from the scattering matrix at fixed energy. We also establish the lead term of the asymptotics for the limiting absorption principle.

Our results use techniques developed by Joshi and Sá Barreto, [17, 18, 19, 20, 21], to study inverse problems in other settings, which build on work by Melrose, [22], and Melrose-Zworski, [23], on the structure of the scattering matrix on asymptotically Euclidean spaces. As in those inverse results, the fundamental

idea here is to compute the symbol of the scattering matrix by solving transport equations along geodesics on the sphere at infinity. These equations express the propagation of growth at infinity.

The analysis here is, however, considerably more involved as the unperturbed wave speed c_0 is not smooth on the compactified space achieved by adding the sphere at infinity, even when $c_0(y)$ is a smooth function of y. This is because c_0 does not have nice asymptotics in $|z|$. The upshot of this is that c_0 is well-behaved on the compactified space only after the space has been blown-up on the equator at infinity. This manifests itself in our analysis by requiring the geodesic flow at infinity to be refracted and reflected by the equator. It was also seen in [5] that it makes the asymptotics in the limiting absorption principle much more complicated. There is a certain similarity here with many-body scattering, compare, e.g. [25]. There the scattering problem is complicated by the presence of a potential that does not decay in certain directions and thus appears as a spike on the sphere at infinity which causes refractions and reflections of the geodesic flows, [25]. Indeed, the case where $c_+ = c_-$ bears much resemblance to the many-body case. However, when c_+ and c_- are different, there are effectively different energy levels in the two hemi-spheres, which introduces new complications not present in the many-body setting, and much of this paper is dedicated to coping with those complications.

In Section 4.2 we define the scattering matrix, and its "main part." In case the operator $c_0^2(D_y^2 + \rho^2)$ has no eigenvalues as an operator on $L^2(\mathbb{R}, c_0^{-2}dy)$, then the main part of the scattering matrix is the same as the scattering matrix.

Our first main result is

Theorem 1.1. *Suppose c, c_0 satisfy the general assumptions of Section 2, and either hypothesis (H1) or (H2). Then, if $c_+ = c_-$, the main part of the scattering matrix is a zeroth order Fourier integral operator associated with broken geodesic flow at time π. If $c_- > c_+$, then the main part of the scattering matrix is a sum of Fourier integral operators associated with the mapping*

$$(\overline{\omega}, \omega_n) \mapsto (-\overline{\omega}, \omega_n)$$

and the mapping

$$(\overline{\omega}, \omega_n) \mapsto (-c_-\overline{\omega}/c_+, -\sqrt{1 - c_-^2|\overline{\omega}|^2/c_+^2})$$

if $\sqrt{1 - c_+^2/c_-^2} < \omega_n$ and

$$(\overline{\omega}, \omega_n) \mapsto (-c_+\overline{\omega}/c_-, \sqrt{1 - c_+^2|\overline{\omega}|^2/c_-^2})$$

if $\omega_n < 0$.

Here, when $c_+ = c_-$, the geodesic flow is broken at the equator $(\overline{\omega}, 0) \subset \mathbb{S}^{n-1}$. This can be compared to the situation for the Laplacian ([23]), or a perturbation of the Laplacian to an integral power ([6]), on a manifold with asymptotically Euclidean ends, where the scattering matrix is a zeroth order Fourier integral operator associated to geodesic flow at time π on the the boundary "at infinity." An additional analogy is to 3-body scattering, where the three-cluster to three-cluster part of the scattering matrix is a sum of Fourier integral operators associated to broken geodesic flow at time π ([25]). Other results on the structure of the scattering matrix in n-body scattering may be found in [26]

Further results on the structure of the scattering matrix are given in Proposition 6.4.

Our central inverse result is

Theorem 1.2. *Suppose c and c_0 satisfy the general assumptions of Section 2, as well as either hypothesis (H1) or (H2), and $n \geq 3$. Then, if $c_+ = c_-$, the asymptotic expansion at infinity of $c - c_0$ is uniquely determined by c_0 and the transmitted singularities of the main part of the scattering matrix at fixed nonzero energy. If $c_+ < c_-$, then the asymptotic expansion is uniquely determined by c_0 and the reflected singularities of the main part of the scattering matrix at fixed nonzero energy.*

The reflected singularities are those associated to the mapping $(\overline{\omega}, \omega_n) \mapsto (-\overline{\omega}, \omega_n)$ and, for $c_+ = c_-$, the transmitted singularities are those associated to the mapping $\omega \mapsto -\omega$. Corollary 7.1 shows that knowledge

of c_+, c_-, and the singularities of the scattering matrix at fixed nonzero energy determine c, within the class we consider, modulo a function vanishing faster than the reciprocal of any polynomial at infinity.

Following the approach to studying the scattering matrix introduced in [23], in Section 5 we construct a parametrix for the Poisson operator. This is a key part of our proofs, as it facilitates an understanding of the singularities of the scattering matrix. We work particularly by adapting the techniques of [18] which are essentially a concretization of the approach introduced in [23]. However, the different behaviour of the unperturbed operator $c_0^2\Delta$ in different regions at infinity means that the analysis is considerably more involved.

To pass from a parametrix to the actual Poisson operator, we need a good understanding of the behaviour of $(\Delta-(\lambda-i0)^2c^{-2})^{-1}f$ at infinity, when $f\in\langle z\rangle^{-\infty}L^2(\mathbb{R}^n)$ and $(1-\phi(y))f\in\mathcal{S}(\mathbb{R}^n)$ for some $\phi\in C_c^\infty(\mathbb{R})$. In practice, the f for which we apply this will be the error from the parametrix of the Poisson operator. When $c_+=c_-$, we can do this by modifying some n-body results of [13] and [24]. However, when $c_+<c_-$ these results no longer apply, and we develop new techniques. The essential idea of these techniques is to repeatedly develop better approximations with improving smoothness properties. Thus Section 6 is devoted to understanding $(\Delta-(\lambda-i0)^2c^{-2})^{-1}$, allowing us to finish the proof of Theorem 1.1. In particular, we prove the following limiting absorption principle

Theorem 1.3. *Let c and c_0 satisfy the hypotheses of Section 2 and hypothesis (H1) or (H2). For any $\chi\in C_c^\infty(\mathbb{S}_c^{n-1})$, $f\in\langle z\rangle^{-\infty}L^2(\mathbb{R}^n)$, $(1-\phi(y))f\in\mathcal{S}(\mathbb{R}^n)$ for some $\phi\in C_c^\infty(\mathbb{R})$, we have*

$$\chi(z/|z|)(\Delta-c^{-2}(\lambda-i0)^2)^{-1}f=e^{-i\lambda|z|/c}|z|^{-(n-1)/2}a_0(z/|z|)+u_1$$

where $a_0\in C^\infty(\mathbb{S}_c^{n-1})$ and $u_1\in\langle z\rangle^\epsilon L^2(\mathbb{R}^n)$ for all $\epsilon>0$.

Here $C_c^\infty(\mathbb{S}_c^{n-1})$ is the space of smooth functions vanishing in a neighbourhood of the equator and in a neighbourhood of $\{(\overline{\omega},\omega_n)\in\mathbb{S}^{n-1}:\omega_n=\sqrt{1-c_+^2/c_-^2}\}$.

In Section 7, we use a modification of some techniques of [19] to prove Theorem 1.2 and apply some one-dimensional scattering theory to give further inverse results.

An announcement of some of these results and an outline of part of the proof can be found in the lecture notes [7].

We are grateful to Fritz Gesztesy for helpful conversations and providing useful references and to Jim Ralston for helpful discussions. We thank the London Mathematical Society for supporting this collaborative research through its small grants scheme. The first author is grateful to the N.S.F. for partial support.

2. Assumptions and Notation

Throughout, $z=(x,y)\in\mathbb{R}^{n-1}\times\mathbb{R}$.

Both sound speeds c and c_0 satisfy $0\le c_m<c,c_0<c_M<\infty$. Moreover, $c_0(y)$ is piecewise smooth and there exists a finite y_M so that $c_0(y)=c_\pm$ when $\pm y>y_M$, with $c_-\ge c_+$. Moreover, all derivatives of c_0 are bounded except at finitely many values of y. This allows c_0 to be piecewise constant, for example.

We require that, away from the hypersurface $\{y=0\}$, $c-c_0$ be smooth outside of a compact set K, and for simplicity we choose y_M so that $K\subset\mathbb{R}^{n-1}\times[-y_M,y_M]$. Moreover, we make requirements on the behaviour of $c-c_0$ at infinity. We have, for $y\neq0$,

$$D_z^\alpha(c(z)-c_0(y))=D_z^\alpha\sum_{j\ge J}^{N}\gamma_j(\frac{z}{|z|})|z|^{-j}+\mathcal{O}(|z|^{-N-1-|\alpha|})\tag{4}$$

for any N and any multiindex α, where $\gamma_j\in C_b^\infty(\mathbb{S}^{n-1}\setminus\{(\overline{\omega},0)\})$. Here we use the notation that $C_b^\infty(X)$ is the space of smooth functions on X that have all derivatives bounded. We shall take J at least 2 everywhere, although sometimes we shall require it to be larger. Some of our results hold under less restrictive hypotheses.

Additionally, we shall often use one of the following hypotheses:
(H1) $J = 2$, $c_+ = c_-$, c and c_0 are smooth.
(H2) $J \geq 4$.

We warn the reader that the choice of the total space dimension to be n rather than $n+1$ is in disagreement with [5] and many other papers on the subject.

We use the notation $\langle w \rangle = (1 + |w|^2)^{1/2}$. Throughout, ϵ shall stand for a small positive quantity and C for a positive constant, either of which may change from line to line.

3. Spectral theory of $c_0^2\Delta$

In order to define the (absolute) scattering matrix for $c^2\Delta$, we will need some understanding of the generalized eigenfunctions of $c_0^2\Delta$ and $c^2\Delta$, particularly of the space that parameterizes them. Further details can be found in, for example, [3, 12, 28, 30].

The operators $c_0^2\Delta$ and $c^2\Delta$ are formally self-adjoint on $L^2(\mathbb{R}^n, c_0^{-2}dz)$ and $L^2(\mathbb{R}^n, c^{-2}dz)$, respectively and have a unique self-adjoint extension.

Roughly speaking, the spectral measure of $c_0^2\Delta$ can be given in terms of two kinds of families of functions. At fixed energy λ, the first is parameterized by $\mathbb{S}_c^{n-1}$, almost as for the Laplacian, though the generalized eigenfunctions are more complicated. Here $\mathbb{S}_c^{n-1} = \{\omega = (\overline{\omega}, \omega_n) \in \mathbb{S}^{n-1} : \omega_n \neq 0,\ \omega_n \neq \sqrt{1 - c_+^2/c_-^2}\}$. (Compare [28, Section 2.1].)

A second type of generalized eigenfunction comes from eigenvalues of $c_0^2(\kappa^2 + D_y^2)$ on $L^2(\mathbb{R}, c_0^{-2}dy)$, if there are any. If there are any eigenvalues, let $\lambda_1^2(\kappa) < \lambda_2^2(\kappa) < \cdots < \lambda_{k(\kappa)}^2(\kappa) < c_+^2\kappa^2$ denote the eigenvalues of $c_0^2(\kappa^2 + D_y^2)$. There may not be any eigenvalues, but if there are, there are only finitely many for fixed κ and the number grows with κ^2. Additionally, if $\kappa > 0$ and $\lambda_j > 0$, then $\frac{d\lambda_j}{d\kappa} > 0$, as can be seen by an integration by parts argument (see, e.g., [5, Sect. 2.2]).

Let κ_j^0 be the smallest positive number such that $c_0^2(\kappa^2 + D_y^2)$ has j eigenvalues for all $\kappa > \kappa_j^0$. Let κ_j be the inverse of λ_j (with the same sign), and let $t_j = \lim_{\kappa \downarrow \kappa_j^0} \lambda_j^2(\kappa) = c_+^2(\kappa_j^0)^2$. The $\{t_j\}$ are called thresholds of $c_0^2\Delta$. Let $T(\lambda)$ be the number of thresholds t_j less than λ^2. For $0 < j \leq T(\lambda)$, $\overline{\omega} \in \mathbb{S}^{n-1}$, let

$$\Phi_j(z, \lambda, \omega) = e^{i\kappa_j(\lambda)x\cdot\overline{\omega}} f_j(y),$$

where $f_j(y) \in L^2(\mathbb{R}, c_0^{-2}dy)$ satisfies

$$c_0^2(\kappa_j^2 + D_y^2) f_j = \lambda^2 f_j$$

and note that $(c_0^2\Delta - \lambda^2)\Phi_j = 0$.

At energy level λ^2, we can parameterize the generalized eigenfunctions of $c_0^2\Delta$ by $\mathbb{S}_c^{n-1}$ and $T(\lambda)$ copies of $\mathbb{S}^{n-2}$. The continuous spectrum of $c^2\Delta$ is parameterized by the same space as that of $c_0^2\Delta$.

Because of the described parametrization of the continuous spectrum at fixed energy, the (absolute) scattering matrices of $c_0^2\Delta$ and $c^2\Delta$ are operators from $L^2(S_c) \oplus_{1 \leq j \leq T(\lambda)} L^2(\mathbb{S}^{n-2})$ into itself. In [5] a definition of the scattering matrix is given in terms of the generalized eigenfunctions. Here, however, it will be more useful to define the (absolute) scattering matrix using the Poisson operator, which we shall do in Section 4.2.

4. The Poisson operator and the scattering matrix

The Poisson operator is defined as an operator

$$P(\lambda) : C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_{i=1}^{T(\lambda)} C^\infty(\mathbb{S}^{n-2}) \to \langle z \rangle^{-1/2-\epsilon} L^2(\mathbb{R}^n).$$

Definition 4.1. *If $g = (g_0, g_1, ..., g_{T(\lambda)}) \in C_c^\infty(\mathbb{S}_c) \oplus_{i=1}^{T(\lambda)} C^\infty(\mathbb{S}^{n-2})$, then $P(\lambda)g = u$, $(c^2\Delta - \lambda^2)u = 0$, and u is determined by its asymptotics at infinity:*

$$u \sim |z|^{-(n-1)/2} e^{i\lambda|z|/c} g_0\left(\frac{z}{|z|}\right) + |x|^{-(n-2)/2} \sum_{j=1}^{T(\lambda)} e^{i\kappa_j(\lambda)|x|} f_j(y) g_j\left(\frac{x}{|x|}\right) + u_0 + \sum_{j=1}^{T(\lambda)} u_j + \tilde{u}.$$

The functions u_0, u_j, $\tilde{u}$ satisfy

$$\begin{gathered}(\tfrac{\partial}{\partial |z|} + i\lambda/c)u_0 \in \langle z\rangle^\epsilon L^2(\mathbb{R}^n);\quad u_0 \in \langle y\rangle^{1/2+\epsilon}\langle z\rangle^\epsilon L^2(\mathbb{R}^n);\\ u_j = |x|^{-(n-2)/2} e^{-i\kappa_j(\lambda)|x|} b_j(x/|x|) f_j(y),\ 1 \le j \le T(\lambda);\\ \tilde{u},\ \tfrac{\partial}{\partial|z|}\tilde{u} \in \langle z\rangle^\epsilon L^2(\mathbb{R}^n),\end{gathered}$$

for any $\epsilon > 0$. Here κ_j, f_j are as in Section 3.

Proposition 4.2 shows that this expansion uniquely determines the Poisson operator. Definition 4.3, using Proposition 4.3, defines the (absolute) scattering matrix via the Poisson operator. The existence of the Poisson operator is proved in Section 6.

4.1. **The Poisson operator is uniquely determined.** In order to show that the Poisson operator above is indeed well-defined, we shall need a uniqueness result, for whose proof we shall use Proposition 4.1.

In proving the following proposition, we shall use some results of Weder, [27, 28] (See also [10].). We recall some of his results below.

Let $A = (-i/4)(z \cdot \nabla_z + \nabla_z \cdot z)$. We define the commutator $[\Delta - \lambda^2/c^2, A]$ as a quadratic form (See the proof of Theorem 5.4, [28].). By [27, Lemma 3.1] for all $\lambda > 0$, $\mu > -\lambda^2/c_-^2$, there is a compact operator K, a compact interval Λ containing μ, and $\beta > 0$ such that

$$iE_\Lambda[\Delta - \lambda^2 c^{-2}, A]E_\Lambda \ge \beta E_\Lambda + K \tag{5}$$

where $E_\Lambda = E_\Lambda(\Delta - \lambda^2 c^{-2})$ is the spectral projector for $\Delta - \lambda^2 c^{-2}$.

The following proposition and its proof, included for the convenience of the reader, are essentially adapted from [2, Lemma 4.17].

Proposition 4.1. *If $u \in \langle z\rangle^\epsilon L^2(\mathbb{R}^n)$ for every $\epsilon > 0$ and $(\Delta - \lambda^2/c^{-2})u = 0$, then $u \equiv 0$.*

Proof. By the results of [27, 28], there is no nontrivial L^2 null space of $\Delta - \lambda^2/c^2$, so it suffices to show that $u \in L^2(\mathbb{R}^n)$.

For $\epsilon, \delta > 0$, let $u_\delta = (1 + \delta\langle z\rangle)^{-\epsilon} u \in L^2(\mathbb{R}^n)$. Let $L = \Delta - \lambda^2/c^2$, and let $\Phi \in C_c^\infty(\mathbb{R})$ be 1 in a neighbourhood of 0.

Note that

$$L\Phi(L)u_\delta = \Phi(L)\left(\sum_j \frac{2\epsilon\delta z_j}{\langle z\rangle}(1 + \delta\langle z\rangle)^{-1}\frac{\partial}{\partial z_j}u_\delta + f_{\epsilon\delta}(z)u\right)$$

where

$$|f_{\epsilon\delta}(z)| \le C\frac{\epsilon}{\langle z\rangle^2}(1 + \delta\langle z\rangle)^{-\epsilon} \tag{6}$$

and the constant C is independent of ϵ and δ.

Then

$$\begin{aligned}([L, A]\Phi(L)u_\delta, \Phi(L)u_\delta) &= -2i\,\mathrm{Im}(AL\Phi(L)u_\delta, \Phi(L)u_\delta)\\ &= -2i\,\mathrm{Im}(A\Phi(L)(\sum_j \frac{2\epsilon\delta z_j}{\langle z\rangle}(1 + \delta\langle z\rangle)^{-1}\frac{\partial}{\partial z_j}u_\delta + f_{\epsilon\delta}(z)u), \Phi(L)u_\delta).\end{aligned} \tag{7}$$

Using this equality, (6), and the fact that

$$\Phi(L) : \langle z\rangle^{-\gamma}L^2(\mathbb{R}^n) \to \langle z\rangle^{-\gamma}L^2(\mathbb{R}^n),$$

we obtain

$$|([L,A]\Phi(L)u_\delta,\Phi(L)u_\delta)|\le \epsilon C_\Phi\|u_\delta\|\|\Phi(L)u_\delta\|+C_2. \tag{8}$$

Here and below C_Φ, C_2 are constants which may change, independent of ϵ and δ, but depending on Φ, and C_2 depends on $\|\langle z\rangle^{-\epsilon_0}u\|$ as well. Since $(1-\Phi(L))u=0$, we have

$$\|u_\delta\|=\|\Phi(L)u_\delta+[\Phi(L),(1+\delta\langle z\rangle)^{-\epsilon}]u\|\le\|\Phi(L)u_\delta\|+C_2. \tag{9}$$

However, using (5) and the fact that 0 is not an eigenvalue of L, we obtain

$$([L,A]\Phi(L)u_\delta,\Phi(L)u_\delta)\ge\beta_1(\Phi(L)u_\delta,\Phi(L)u_\delta)$$

for some $\beta_1>0$, if the support of Φ is chosen sufficiently small.

By choosing ϵ sufficiently small, then, and using (8) and (9), we get that

$$\beta_1\|\Phi(L)u_\delta\|^2\le\frac{\beta_1}{2}\|u_\delta\|^2+C_2.$$

Then

$$\|\Phi(L)u_\delta\|\le C_2,$$

and using (9), this shows that for sufficiently small ϵ, $\|u_\delta\|$ is bounded by a constant independent of δ, and thus $u\in L^2(\mathbb{R}^n)$, and $u\equiv0$. $\square$

We shall use the following notion of an "outgoing" function.

Definition 4.2. *A function $u\in\langle z\rangle^{1/2+\epsilon}L^2(\mathbb{R}^n)$ will be called outgoing if it has a decomposition $u=u_0+\sum_1^{T(\lambda)}u_j+\tilde u$ with the following properties*

$$\begin{gathered}(\tfrac{\partial}{\partial|z|}+i\lambda/c)u_0\in\langle z\rangle^\epsilon L^2(\mathbb{R}^n);\;\; u_0\in\langle y\rangle^{1/2+\epsilon}\langle z\rangle^\epsilon L^2(\mathbb{R}^n);\\ u_j=|x|^{-(n-2)/2}e^{-i\kappa_j(\lambda)|x|}b_j(x/|x|)f_j(y),\ 1\le j\le T(\lambda);\\ \tilde u,\tfrac{\partial}{\partial|z|}\tilde u\in\langle z\rangle^\epsilon L^2(\mathbb{R}^n),\end{gathered}$$

for any $\epsilon>0$.

Proposition 4.2. *Given $f\in L^2(\mathbb{R}^n)$, there is at most one outgoing $u\in\langle z\rangle^{1/2+\epsilon}L^2(\mathbb{R}^n)$ with $(\Delta-\lambda^2/c_0^2)u=f$.*

Proof. Suppose there are two such u. Then by considering the difference we can reduce this to the case $f\equiv0$. Then

$$\begin{aligned}0&=\int_{|z|<R}(\Delta-\lambda^2/c^2)u\overline{u}=-\int_{|z|=R}(\frac{\partial}{\partial|z|}u\overline{u}-u\frac{\partial}{\partial|z|}\overline{u})\\&=2\int_{|z|=R}\frac{i\lambda}{c}|u_0|^2+\sum_{j,k}i\kappa_j(\lambda)u_j\overline{u}_k+i\operatorname{Re}\sum_j(\lambda/c+\kappa_j)u_0\overline{u}_j+i\operatorname{Im}(u_0e_0+\sum_ju_je_j)+i\operatorname{Im}\tilde e\tilde f\end{aligned} \tag{10}$$

where $e_0,e_j,\tilde e,\tilde f\in\langle z\rangle^\epsilon L^2(\mathbb{R}^n)$ for all $\epsilon>0$. Using the facts that $\int f_j(y)\overline{f}_k(y)dy=0$ if $j\ne k$ and f_j is exponentially decreasing, this implies that as $R\to\infty$

$$\int_{|z|=R}\frac{2i\lambda}{c}|u_0|^2+\sum_j2i\kappa_j(\lambda)|u_j|^2\le C\int_{|z|=R}(\sum_j|u_0\overline{u}_j|+|u_0||e_0|+\sum_j|u_je_j|+|\tilde e||\tilde f|)+\mathcal{O}(R^{-2-\epsilon}). \tag{11}$$

Since $u_j\in\langle y\rangle^{-\infty}\langle x\rangle^{1/2+\epsilon}L^2(\mathbb{R}^n)$ and $u_0\in\langle y\rangle^{1/2+\epsilon}\langle z\rangle^\epsilon L^2(\mathbb{R}^n)$, we have $u_0\overline{u}_j\in\langle y\rangle^{-\infty}\langle z\rangle^{1/2+2\epsilon}L^1(\mathbb{R}^n)$. Therefore, the right hand side of (11), considered as a function of R for large R, is in $R^{1/2+2\epsilon}L^1(\mathbb{R}_+)$, so

that $u_0,\ u_j \in \langle z\rangle^{1/4+\epsilon}L^2(\mathbb{R}^n)$. This in turn means that $b_j = 0 = u_j$ for $1 \leq j \leq T(\lambda)$. Now we have $\frac{\partial}{\partial|z|}u = (-i\lambda/c)u + \tilde{u}$, $\tilde{u} \in \langle z\rangle^{\epsilon}L^2(\mathbb{R}^n)$, any $\epsilon > 0$. Repeating the argument almost as above, we obtain

$$\int_{|z|=R}|u|^2 \leq C\int_{|z|=R}|u\tilde{u}| \leq C(\int_{|z|=R}|u|^2)^{1/2}(\int_{|z|=R}|\tilde{u}|^2)^{1/2}$$

and thus

$$(\int_{|z|=R}|u|^2)^{1/2} \leq C(\int_{|z|=R}|\tilde{u}|^2)^{1/2}$$

so that $u \in \langle z\rangle^{\epsilon}L^2(\mathbb{R}^n)$, any $\epsilon > 0$. By the previous Proposition, $u \equiv 0$. $\square$

4.2. **The Absolute Scattering Matrix.** In order to use the Poisson operator to define the scattering matrix, we shall need the following proposition, whose proof is a corollary of our construction of Section 5 and Theorem 1.3 (see Section 6.2).

Proposition 4.3. *Let $g = (g_0, g_1, ..., g_{T(\lambda)}) \in C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} C^\infty(\mathbb{S}^{n-2})$. Let $\theta = \frac{z}{|z|} \in K$, for K a compact set in $\mathbb{S}_c^{n-1}$. Then, for $\theta \in K$, as $|z| \to \infty$,*

$$P(\lambda)g = |z|^{-(n-1)/2}[e^{i\lambda|z|/c}g_0(\theta)_{|K} + e^{-i\lambda|z|/c}(g_0'(\theta))_{|K}] + \tilde{u}_K$$

where $\tilde{u}_K \in \langle z\rangle^{\epsilon}L^2(\mathbb{R}^n \cap (K \times [1,\infty)))$ for any $\epsilon > 0$.
Let $y \in K_y$, $K_y \subset \mathbb{R}$ compact, and let $\overline{\theta} = \frac{x}{|x|}$. Then, as $|x| \to \infty$,

$$P(\lambda)g = |x|^{-(n-2)/2}[\sum_1^{T(\lambda)} e^{i\kappa_j(\lambda)|x|}g_j(\theta)f_j(y)_{|K_y} + \sum_1^{T(\lambda)} e^{-i\kappa_j(\lambda)|x|}g_j'(\theta)f_j(y)_{|K_y}] + \tilde{u}_c$$

where $\tilde{u}_c \in \langle x\rangle^{\epsilon}L^2((\mathbb{R}^n \cap (\{|x| > 1\}) \times K_y))$ for any $\epsilon > 0$.

This information about the Poisson operator allows us to define the (absolute) scattering matrix $A(\lambda)$.

Definition 4.3. *The (absolute) scattering matrix $A(\lambda)$ is given, for $g \in C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} C^\infty(S^{n-2})$, by $A(\lambda)g = g' \in L^2(\mathbb{S}_c^{n-1}) \oplus L^2(S^{n-2})$, where for any compact set $K \subset \mathbb{S}_c^{n-1}$, $(g_0')_{|K}$ is as in Proposition 4.3, and, for $1 \leq j \leq T(\lambda)$, g_j' is as in Proposition 4.3.*

We remark that this definition differs slightly from the absolute scattering matrix discussed in [5]. However, as the two differ by a straight-forward normalization, we shall use this definition here both to emphasize the similarities with the absolute scattering matrix as defined in [22] and because it is most convenient for the inverse results.

For fixed λ, $A(\lambda) = (A_{ij}(\lambda))$, $0 \leq i, j \leq T(\lambda)$, with the A_{ij} operators. We call $A_{00}(\lambda)$ the "main part" of the scattering matrix. If the operator $c_0^2(D_y + \rho^2)$ has no eigenvalues on $L^2(\mathbb{R}, c_0^{-2}dy)$, the "main part" of the scattering matrix is just the scattering matrix.

5. The approximate Poisson operator

For $g = (g_0, g_1, ..., g_{T(\lambda)}) \in L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2})$, let $\Pi_j g = g_j$. Since our inverse results involve the main part of the scattering matrix, $A_{00}(\lambda)$, we are most interested in $P_0(\lambda) = P(\lambda)\Pi_0$, where $P(\lambda)$ is the Poisson operator. Here we construct an approximation $\tilde{P}_0$ of P_0.

Let $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and let $\omega \in \mathbb{S}^{n-1}$, $\omega = (\overline{\omega}, \omega_n)$, $\omega_n \neq 0$. We will construct a (partial) approximate Poisson operator $\tilde{P}_0(z, \lambda, \omega)$ such that

$$(c^2\Delta - \lambda^2)\tilde{P} \in \langle z\rangle^{-\infty}L^2(\mathbb{R}^n),\ (1 - \chi(y))(c^2\Delta - \lambda^2)\tilde{P}_0 \in \mathcal{S}(\mathbb{R}^n)$$

where $\chi(y) \in C_c^\infty(\mathbb{R})$ is 1 for $|y| \leq y_M + 1$, and, so that distributionally as $|z| \to \infty$,

$$\tilde{P}_0(z, \lambda, \omega) = |z|^{-(n-1)/2}\left(e^{i\lambda|z|/c_0(y)}\delta_\omega(\frac{z}{|z|}) + e^{-i\lambda|z|/c_0(y)}g(\omega, \frac{z}{|z|})\right) + \mathcal{O}(|z|^{-(n+1)/2}).$$

We will show how to construct such an approximation when $\omega_n > 0$; the case of $\omega_n < 0$ is quite similar. The construction involves solving away errors at infinity. Since the model operator $c_0^2\Delta$ has different behaviour depending on the region "at infinity" ($y > y_M$, $|y| < y_M$, or $y < -y_M$) the techniques involved necessarily depend on the region in which z lies.

We shall need the following

Lemma 5.1. *For $f \in L^2([-y_M, y_M])$, the boundary value problem*

$$c_0^2(\lambda^2|\overline{\omega}|^2/c_+^2 + D_y^2)b - \lambda^2 b = f$$

with boundary conditions

$$\begin{aligned} -i\lambda\omega_n/c_+ b(y_M) - b'(y_M) &= \alpha_1 \\ i\lambda(1/c_-^2 - |\overline{\omega}|^2 1/c_+^2)^{1/2} b(-y_M) - b'(-y_M) &= \alpha_2 \end{aligned}$$

has a unique solution in $L^2([0, y_0], c_0^{-2}dy)$ if $1/c_-^2 \geq |\overline{\omega}|^2/c_+^2$.

Proof. This boundary problem can be reduced to the form

$$\begin{aligned} -i\lambda\omega_n/c_+ b(y_M) - b'(y_M) &= 0 \\ i\lambda(1/c_-^2 - |\overline{\omega}|^2/c_+^2)^{1/2} b(-y_M) - b'(-y_M) &= 0 \\ c_0^2(\lambda^2|\overline{\omega}|^2/c_+^2 + D_j^2)b - \lambda^2 b &= g. \end{aligned}$$

This has a solution if the adjoint operator has no nontrivial null space; and the solution is unique if the homogeneous equation has no nontrivial solutions.

The adjoint operator is the operator

$$c_0^2(\lambda^2|\overline{\omega}|^2/c_+^2 + D_j^2) - \lambda^2$$

with domain

$$\begin{aligned} \{h \in L^2([0, y_0], c_0^{-2}dy) : -i\lambda\omega_n/c_+ h(y_M) + h'(y_M) = 0 \\ \text{and } i\lambda(1/c_-^2 - |\overline{\omega}|^2/c_+^2)^{1/2} h(-y_M) + h'(-y_M) = 0\}. \end{aligned} \tag{12}$$

Suppose g is a nontrivial element of the null space of the adjoint operator. Then

$$\begin{aligned} 0 &= \int_{-y_M}^{y_M} (c_0^2(\lambda^2|\overline{\omega}|^2/c_+^2 + D_y^2) - \lambda^2) g\overline{g} c_0^{-2} dy \\ &= -g'(y_M)\overline{g}(y_M) + g'(-y_M)\overline{g}(-y_M) + g(y_M)\overline{g}'(y_M) - g(-y_M)\overline{g}'(-y_M) \\ &+ \int_{-y_M}^{y_M} \left[(\lambda^2|\overline{\omega}|^2/c_+^2 - \lambda^2 c_0^{-2})|g|^2 + |D_y g|^2\right] dy \end{aligned} \tag{13}$$

Using the boundary conditions, we find that the second line is equal to

$$-2i\lambda\omega_n/c_+|g(y_M)|^2 - 2i\lambda(1/c_-^2 - |\overline{\omega}|^2/c_+^2)^{1/2}|g(-y_M)|^2.$$

Since the third line of (13) is real, this means that $0 = g(y_M) = g'(y_M)$, or $g(y) \equiv 0$.

A similar calculation shows that the original operator has no nontrivial null space. ∎

Let $\omega = (\overline{\omega}, \omega_n)$ with $\omega_n > 0$. In our construction of the approximation to the Poisson operator P_0, we begin with the function $\Phi(z, \lambda, \omega)$ which is defined by

$$e^{i\lambda x\cdot\overline{\omega}/c_+}\phi_+(y)$$

where ϕ_+ satisfies

$$c_0^2(D_y^2 + \lambda^2(1 - \omega_n^2)c_+^{-2})\phi_+ = \lambda^2\phi_+ \tag{14}$$

and, as $y \to \infty$,

$$\phi_+(y) \sim e^{i\lambda y\omega_n/c_+} + R_+(\lambda,\omega_n)e^{-i\lambda\omega_n y/c_+} \tag{15}$$

and as $y \to -\infty$,

$$\phi_+(y) \sim T_+(\lambda,\omega_n)e^{i\lambda y\sqrt{1/c_-^2-1/c_+^2+\omega_n^2/c_+^2}} \tag{16}$$

where when $1/c_-^2 - 1/c_+^2 + \omega_n^2/c_+^2 < 0$ we take the square root so that the right hand side of (16) is exponentially decreasing. We have $(c_0^2\Delta - \lambda^2)\Phi = 0$. Note that, up to a constant multiple which depends only on n, λ, and $c_\pm$, Φ is the Schwartz kernel of the (partial) Poisson operator $P_{0,0}$ for $c_0^2\Delta$ when ω is in the upper hemisphere of $\mathbb{S}^{n-1}$. We use this as our starting point.

When $y > y_M$, we use the techniques of [18] to construct $\tilde{P}$. Note that when we apply $c^2\Delta - \lambda^2$ to Φ we obtain an error which, for $y > y_M$, is of the form

$$a_1 e^{i\lambda z\cdot\omega/c_+} + a_2 e^{i\lambda(x\cdot\overline{\omega}-y\omega_n)/c_+} \tag{17}$$

where a_1, a_2 are classical polyhomogeneous symbols (in $|z|$) of order -2. We use the techniques of [18] to find a term of the form $ae^{i\lambda z\cdot\omega/c_+}$ with $a \in S^{-1}_{phg}$ which will solve away the first term in the error (17) here. Just as in [18], this is done iteratively, solving away an error $d \in e^{i\lambda z\cdot\omega/c_+}(S^{-j}_{phg}\mathrm{mod}S^{-j-1}_{phg})$ by using $be^{i\lambda z\cdot\omega/c_+}$ with $b \in S^{-j+1}_{phg}$, and b solving the transport equation

$$-2ic_+\lambda\omega\cdot\frac{\partial b}{\partial z} = d \tag{18}$$

along the geodesics on the unit sphere at infinity. We choose b so that b is smooth at $z/|z| = \omega$ in order to keep the right coefficient of $e^{i\lambda|z|/c_+}$ in the distributional asymptotic expansion.

Let s be the geodesic distance on the sphere from ω and let $\tilde{\theta}$ be angular coordinates about ω. The equation (18) can be solved, modulo an error of lower order, by $b_{I,-j+1}|z|^{-j+1}$ just as in [18, Section 2], giving

$$b_{I,-j+1}(s,\tilde{\theta},\omega) = \frac{i}{2\lambda c_+(\sin s)^{j-1}}\int_0^s (\sin s')^{j-2}d_{-j,I}(s',\tilde{\theta};\omega)ds' \tag{19}$$

where $d_{-j,I}|z|^{-j}$ is the term of homogeneity $-j$ in the error. Note that as long as $z/|z|$ is in the upper hemisphere we are away from $-\omega$ so the transport equations have a smooth solution.

We find $b_{I,-j}$ iteratively and then use Borel's lemma to asymptotically sum them, obtaining a b_I such that

$$(c^2\Delta - \lambda^2)(\Phi + b_I e^{i\lambda z\cdot\omega/c_+}) = e^{i\lambda x\cdot\overline{\omega}/c_+}e^{-i\lambda y\omega_n/c_+}a_2 + \mathcal{O}(|z|^{-\infty})$$

when $y > y_M$. Note that the construction of b_I has not changed a_2.

We will apply almost the same technique to solve away the error $e^{i\lambda x\cdot\overline{\omega}/c_+}e^{-i\lambda y\omega_n/c_+}a_2$, $y > y_M$, away from $z/|z| = (-\overline{\omega}, \omega_n)$. Here we will use solutions to the transport equation where we choose the initial condition at $y/|z| = 0$, and the solutions, in analogy to (19), are of the form

$$b_{R,-j+1}(s,\tilde{\theta},\omega) = \frac{i}{2\lambda c_+(\sin s)^{j-1}}\left[\int_{s_{R_0}}^s (\sin s')^{j-2}d_{-j,R}(s',\tilde{\theta};\omega)ds' + C_{R,j-1}\right]. \tag{20}$$

Here s is the distance on $\mathbb{S}^{n-1}$ from $\theta = z/|z|$ to the point $(\overline{\omega}, -\omega_n)$ and $\tilde{\theta}$ is the angular coordinate about $(\overline{\omega}, -\omega_n)$. The value $s = s_{R_0}$ corresponds to $\theta_n = 0$, and $C_{R,j}$ depends only on ω and $\tilde{\theta}$. We postpone to Section 5.1 discussion of the form of the parametrix near $z/|z| = (-\overline{\omega}, \omega_n)$.

In the lower hemisphere, we use a similar technique if $1/c_-^2 - |\overline{\omega}|^2/c_+^2 > 0$. Here the error term is of the form

$$e^{i\lambda x\cdot\overline{\omega}/c_+}e^{i\lambda\sqrt{1/c_-^2-|\overline{\omega}|^2/c_+^2}\,y}a_T,$$

where $a_T \in S^{-2}_{phg}$. Again we have solutions like (19) to the transport equation, although this time s measures the distance on the sphere from the point $(c_-\overline{\omega}/c_+, \sqrt{1-c_-^2|\overline{\omega}|^2/c_+^2})$. We will have a solution to the transport equation away from $(-c_-\overline{\omega}/c_+, -\sqrt{1-c_-^2|\overline{\omega}|^2/c_+^2})$ of the form

$$e^{i\lambda x\cdot\overline{\omega}/c_+}e^{i\lambda\sqrt{1/c_-^2-|\overline{\omega}|^2/c_+^2}y}\sum b_{T,-j}|z|^{-j},$$

where

$$b_{T,-j+1}(s,\tilde{\theta},\omega) = \frac{i}{2\lambda c_-(\sin s)^{j-1}}\left[\int_{s_{T_0}}^{s}(\sin s')^{j-2}d_{-j,T}(s',\tilde{\theta};\omega)ds' + C_{T,j-1}\right].$$

The constants (in s) $C_{R,j}$ and $C_{T,j}$ are to be determined. Of course their values affect subsequent errors and thus subsequent $b_{R,-j}$, $b_{T,-j}$.

We will use a different technique to construct the solutions when $|y| < y_M$. We choose the solutions so that $\tilde{P}_0$ is C^1 on $\mathbb{R}^n$. We point out that if c_0 is not smooth, for example, if it is piecewise constant, we should expect it to be impossible to find a smooth Poisson operator on $\mathbb{R}^n$.

The values of $C_{R,j}$, $C_{T,j}$ are determined by solutions to boundary value problems that arise in constructing the parametrix when $|y| < y_M$, as described below.

When $|y| < y_M$, we look for an approximate solution of the form

$$e^{i\lambda x\cdot\overline{\omega}/c_+}\sum_{j\geq 0}|z|^{-j}\left(b_{M,-j}(\frac{x}{|x|},y) + |z|^{-1}\tilde{b}_{M,-j}(\frac{x}{|x|},y)\right). \tag{21}$$

The term $|z|^{-1}\tilde{b}_{M,-j}$ is of lower order and is included to improve the regularity at $y = \pm y_M$. Note that

$$(c^2\Delta - \lambda^2)(|z|^{-j}b(\frac{x}{|x|},y)e^{i\lambda x\cdot\overline{\omega}/c_+}) = \left(c_0^2(D_y^2 + \lambda^2|\overline{\omega}|^2/c_+^2)b - \lambda^2 b\right)e^{i\lambda x\cdot\overline{\omega}/c_+}|z|^{-j} + \mathcal{O}(|z|^{-j-1}).$$

Therefore, for $|y| < y_M$, to solve away an error of the form $|z|^{-j}d_{M,-j}(\frac{x}{|x|},y)$ we look for $b_{M,-j}$ such that

$$c_0^2(D_y^2 + \lambda^2|\overline{\omega}|^2/c_+^2)b_{M,-j} - \lambda^2 b_{M,-j} = d_{M,j}(\frac{x}{|x|},y).$$

The boundary conditions which $b_{M,j}$ must satisfy come from matching with the solutions in the top and bottom hemispheres in order to get a C^1 function. They are

$$\begin{aligned}
b_{I,-j}(x/|x|,0)e^{i\lambda\omega_n y_M/c_+} + e^{-i\lambda\omega_n y_M/c_+}C_{Rj}(x/|x|)(\sin s_{R_0})^{-j}/2i\lambda c_+ &= b_{M,-j}(x/|x|,y_M)\\
i\lambda\omega_n/c_+[b_{I,-j}(x/|x|,0)e^{i\lambda\omega_n y_M/c_+} - e^{-i\lambda\omega_n y_M/c_+}C_{Rj}(x/|x|)(\sin s_{R_0})^{-j}/2i\lambda c_+] &= b'_{M,-j}(x/|x|,y_M)\\
e^{-i\lambda\sqrt{1/c_-^2-|\overline{\omega}|^2/c_+^2}y_M}C_{Tj}(x/|x|)(\sin s_{T_0})^{-j}/2i\lambda c_- &= b_{M,-j}(x/|x|,-y_M)\\
i\lambda(1/c_-^2 - 1/c_+^2|\overline{\omega}|^2)^{1/2}e^{-i\lambda\sqrt{1/c_-^2-|\overline{\omega}|^2/c_+^2}y_M}C_{Tj}(x/|x|)(\sin s_{T_0})^{-j}/2i\lambda c_- &= b'_{M,-j}(x/|x|,-y_M)
\end{aligned} \tag{22}$$

where $b_{I,-j+1}$ is known (it is determined by integrals over portions of geodesics of $(\sin s)^{j-1}d_j$) and C_{Tj}, C_{Rj} are to be determined and are independent of y, and so can be treated as constants in solving the boundary value problem. They can be eliminated from this set of equations, resulting in a boundary value problem of the type considered in Lemma 5.1, which guarantees us a unique solution to the problem when $1/c_-^2 > |\overline{\omega}|/c_+^2$. This then determines $b_{R,-j}$ and $b_{T,-j}$, since $C_{T,j}$ and $C_{R,j}$ are determined by $b_{M,-j}$.

In order to ensure that our function will be C^1 at $y = y_M$ and at $y = -y_M$ we will add an additional term $\tilde{b}_{M,j}$ whose total contribution will be of order $|z|^{-j-1}$. Let $\chi \in C_c^\infty(\mathbb{R})$, $\chi(t) = 1$ for $|t| < 1$ and $\chi(t) = 0$ for

$|t| > 2$. Let

$$\begin{aligned}
\beta_{Uj} &= \lim_{y \downarrow y_M} \left(e^{i\lambda y \omega_n / c_+} (b_{I,-j}(z/|z|) + e^{-i\lambda y \omega_n / c_+} b_{R,-j}(z/|z|) \right) - b_{M,-j}(x/|x|, y_M) \\
\gamma_{Uj} &= \lim_{y \downarrow y_M} \left(\frac{\partial}{\partial y} (e^{i\lambda y \omega_n / c_+} (b_{I,-j}(z/|z|)) + \frac{\partial}{\partial y} (e^{-i\lambda y \omega_n / c_+} b_{R,-j}(z/|z|)) \right) - b'_{M,-j}(x/|x|, y_M) \\
\beta_{Lj} &= \lim_{y \uparrow -y_M} \left(e^{i\lambda y (1/c_-^2 - |\overline{\omega}|^2 / c_+^2)^{1/2}} b_{T,-j}(z/|z|) \right) - b_{M,-j}(x/|x|, -y_M) \\
\gamma_{Lj} &= \lim_{y \uparrow -y_M} \frac{\partial}{\partial y} \left(e^{i\lambda y (1/c_-^2 - |\overline{\omega}|^2 / c_+^2)^{1/2}} b_{T,-j}(z/|z|) \right) - b'_{M,-j}(x/|x|, -y_M)
\end{aligned}$$

Note that by our choice of b_j, β_{Uj}, γ_{Uj}, β_{Lj} and γ_{Lj} all have leading order $|z|^{-1}$. Now, let

$$\tilde{b}_{M,-j} = \left(\chi \left(\frac{3(y - y_M)}{y_M} \right) [\beta_{Uj} + (y - y_M) \gamma_{Uj}] + \chi \left(\frac{3(y - y_M)}{y_M} \right) [\beta_{Lj} + y \gamma_{Lj}] \right) |z|. \tag{23}$$

For $|y| < y_M$, this determines the approximate solution of the form (21).

If $1/c_-^2 - |\overline{\omega}|^2 / c_+^2 < 0$, then we use a slightly different method for finding the approximate solution when $y \le y_M$. Here, in a manner similar to that used for $|y| < y_M$ above, we solve away the error term by using an approximation of the form

$$\sum e^{i\lambda x \cdot \overline{\omega} / c_+} (b_{L,-j}(\frac{x}{|x|}, y) + \tilde{b}_{L,-j}(\frac{x}{|x|}, y) |z|^{-1}) |z|^{-j}$$

where

$$c_0^2 (\lambda^2 |\overline{\omega}|^2 / c_+^2 + D_y^2) b_{L,-j} - \lambda^2 b_{L,-j} = d_{-j},$$

d_{-j} is the coefficient of $|z|^{-j} e^{i\lambda x \cdot \overline{\omega} / c_+}$ in the error term and is exponentially decreasing in y when $y < 0$, and $b_{L,-j}$ is square integrable on $(-\infty, y_M]$. We need in addition a boundary term at $y = y_M$, and this is provided by the first two equations of (22). An argument like that of Lemma 5.1 shows that there is a unique solution to this problem. As in the previous case, $\tilde{b}_{L,-j}$ is chosen to improve the regularity at $y = y_M$.

Remark . We remark that this construction can be carried out, with some minor modifications, for sound speeds $c_1 = c + d$, where $c - c_0$ has an asymptotic expansion of the type (4), and d is supported in $|y| < y_M$, with $d \sim \sum_{j \ge 0} |z|^{-j} d_j(x/|x|, y)$.

5.1. **Approximate Poisson Operator Near its Singularities.** For $\omega_n > 0$, it remains to describe the approximation of the (partial) Poisson operator near $z/|z| = \theta = (-\omega, \omega_n)$ and, if $1 - c_-^2 |\overline{\omega}|^2 / c_+^2 > 0$, near $z/|z| = \theta = (-c_- \overline{\omega} / c_+, -\sqrt{1 - c_-^2 |\overline{\omega}|^2 / c_+^2})$. The approximation in these regions contributes to the scattering matrix. As these two are quite similar, we will concentrate on the first, using the techniques of [18, Section 3].

Let $w = (w_1, ..., w_n) = (w', w_n) \in \mathbb{R}^n$, and rotate the coordinate system so that ω is the north pole. Denote by $I_\tau^{\gamma,\alpha}$ the class of operators whose Schwartz kernel can, near the south pole, be written as a Schwartz function plus a term of the form

$$\int_0^\infty \int \left(\frac{1}{S|w|} \right)^\gamma S^\alpha e^{i\tau (S w' \cdot \mu - \sqrt{1 + S^2} |w|)} a \left(\frac{1}{S|w|}, S, \mu \right) dS d\mu \tag{24}$$

with $a \in C_c^\infty([0, \epsilon) \times [0, \epsilon) \times \mathbb{S}^{n-2})$. From the results of [18], this class is asymptotically complete in γ. Moreover, a stationary phase computation which can be found in [18] shows that away from the south pole this is equivalent to the class of operators whose kernel is of the form $e^{i\tau w_n} b$, with b a polyhomogeneous symbol in $|w|$ (and smooth in $w/|w|$) of order $-\gamma + (n-1)/2$. We recall below some additional facts about the operators $I_\tau^{\gamma,\alpha}$ from [18].

We recall from [18]

Proposition 5.1. *If $u(w,\omega) \in I_\tau^{\gamma,\alpha}$ and $f \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$, then*

$$e^{i\tau|w|} \int u(|w|\theta, \omega) f(\theta, \omega) d\theta d\omega$$

is a smooth symbolic function in $|w|$ of order $-1-\alpha$ and its lead coefficient is $|w|^{-1-\alpha}\langle K, f\rangle$, where K is the pull-back of the Schwartz kernel of a pseudo-differential operator of order $\alpha - \gamma - (n-2)$ by the map $\theta \mapsto -\theta$. The principal symbol of K determines and is determined by the lead term of the symbol, $a(t, S, \mu)$ of u as $S \to 0+$.

¿From [18, Propositions 3.1 and 3.2], we have

Proposition 5.2. *If $u \in I_\tau^{\gamma,\alpha}$ and $V \sim \sum_{j\geq 2} |w|^{-j} v_j(w/|w|)$, then*

$$(\Delta - \tau^2)u \in I^{\gamma+1,\alpha+1}$$

and

$$Vu \in I^{\gamma+2,\alpha+2}.$$

¿From Lemma 3.2 of [18],

Lemma 5.2. *If $u \in I_\tau^{\gamma,(n-3)/2}$, $V \sim \sum_{j\geq 2} |w|^{-j} v_j(w/|w|)$, and $(\Delta + V - \tau^2)u \in I_\tau^{\gamma+1,\frac{n-1}{2}}$, then $(\Delta + V - \tau^2)u \in I_\tau^{\gamma+1,\frac{n+1}{2}}$.*

Again from [18]

Proposition 5.3. *If $u \in I_\tau^{\infty,\alpha} = \cap_\gamma I_\tau^{\gamma,\alpha}$, then $u = e^{-i\tau|w|} f(w)$, with f a classical symbol of order $-\alpha - 1$.*

We proceed as described in [18, Section 3]. Using the first part of the construction, we have an approximation of the Poisson operator that blows up as $\theta = z/|z|$ approaches $(-\overline{\omega}, \omega_n)$ or $(-c_-\overline{\omega}/c_+, -\sqrt{1 - c_-^2|\overline{\omega}|^2/c_+^2})$. Recalling that $\omega_n > 0$, near $\theta = (-\overline{\omega}, \omega_n)$, we find an approximation of the Poisson operator of the form $u\mathcal{R}_+$, where $u \in I_{\lambda/c_+}^{-(n-1)/2,(n-3)/2}$, and $\mathcal{R}_+ : (\overline{\omega}, \omega_\setminus) \mapsto (\overline{\omega}, -\omega_\setminus)$. For θ near $(-c_-\overline{\omega}/c_+, -\sqrt{1 - c_-^2|\overline{\omega}|^2/c_+^2})$, the approximation of the Poisson operator is of the form $u\mathcal{R}_-$, where $u \in I_{\lambda/c_-}^{-(n-1)/2,(n-3)/2}$ and $\mathcal{R}_- : (\overline{\omega}, \omega_\setminus) \mapsto (\rfloor_-\overline{\omega}/\rfloor_+, \sqrt{\infty - \rfloor_-^\in|\overline{\omega}|^\in/\rfloor_+^\in})$. Putting all of this together, we get an approximation $\tilde{P}_0$ to the (partial) Poisson operator with a remainder term $(c^2\Delta - \lambda^2)\tilde{P}_0$ that is in $\langle z\rangle^{-\infty}L^2$, and is Schwartz after multiplication by a function $\phi \in C_b^\infty(\mathbb{R})$ which vanishes for $|y| < y_M + 1$. This error is then solved away by applying $(\Delta - (\lambda - i0)^2c^{-2})^{-1}c^{-2}$, and it is for this that we need the results of Section 6. Taken together, the approximation of the (partial) Poisson operator described in this section, together with the results of Proposition 5.1 and Theorem 1.3, show Theorem 1.1.

5.2. **Approximation of $P(\lambda)\Pi_j$, $1 \leq j \leq T(\lambda)$.** For completeness, we briefly outline how to construct an approximation $\tilde{P}_j(\lambda)$ to $P(\lambda)\Pi_j$, $1 \leq j \leq T(\lambda)$. The approximation will have the properties

$$(c^2\Delta - \lambda^2)\tilde{P}_j \in \langle z\rangle^{-\infty}L^2(\mathbb{R}^n),\ (1 - \phi(y))(c^2\Delta - \lambda^2)\tilde{P}_j \in \mathcal{S}(\mathbb{R}^n)$$

if $\phi \in C_c^\infty(\mathbb{R})$, $\phi(y) = 1$ when $|y| < y_M + 1$, and, distributionally as $|z| \to \infty$,

$$\tilde{P}_j(z, \lambda, \overline{\omega}) = |x|^{-(n-2)/2}\left(e^{i\kappa_j(\lambda)|x|}\delta_{\overline{\omega}}(\frac{x}{|x|})f_j(y) + e^{-i\kappa_j(\lambda)|x|}h(\overline{\omega}, \frac{x}{|x|})f_j(y)\right) + \mathcal{O}(|x|^{-n/2}\langle y\rangle^{-\infty}).$$

Here $\overline{\omega} \in \mathbb{S}^{n-2}$.

For the construction of $\tilde{P}_j$, we begin with $\Phi_j = e^{i\kappa_j(\lambda)x\cdot\overline{\omega}}f_j(y)$, which is, up to a constant multiple depending on n and κ_j, the (jth partial) Poisson operator for $c_0^2\Delta$. We have

$$(c^2\Delta - \lambda^2)\Phi_j \sim \sum_{k\geq 2} |x|^{-k} e^{i\kappa_j(\lambda)x\cdot\overline{\omega}} d_{-k}(x/|x|, y),$$

where d_{-k} is smooth in $x/|x|$, exponentially decreasing in y, and smooth in y when $|y| > y_M$.

To solve away the error with $k = 2$, we write

$$d_{-2}(x/|x|, y) = d_{-2,1}(x/|x|, y) + d_{-2,2}(x/|x|)f_j(y)$$

where $\int d_{-2,1}(x/|x|, y)f_j(y)c_0^{-2}(y)dy = 0$. Since $d_{-2,1}$ is orthogonal to f_j, we can find g_{-2} such that

$$(c_0^2(D_y^2 + \kappa_j^2) - \lambda^2)g_{-2}(x/|x|, y) = d_{-2,1}(x/|x|, y)$$

and use $e^{i\kappa_j(\lambda)x\cdot\overline{\omega}}|x|^{-2}g_{-2}(x/|x|, y)$ to solve away the $e^{i\kappa_j(\lambda)x\cdot\overline{\omega}}|x|^{-2}d_{-2,1}(x/|x|, y)$ error, up to a term vanishing one order faster at infinity. We note that g_{-2} is exponentially decreasing in y, since $d_{-2,1}$ is, and that this term does not contribute anything to the scattering matrix, since distributionally it is $\mathcal{O}(|x|^{-(n+2)/2})$.

To solve away the error $|x|^{-2}e^{i\kappa_j(\lambda)x\cdot\overline{\omega}}d_{-2,2}(x/|x|)f_j(y)$, we use the techniques of [18] in the x variables only. That is, essentially as in our construction of $\tilde{P}_0$, we solve transport equations along geodesics on $\mathbb{S}^{n-2}$ beginning at $x/|x| = \overline{\omega}$. Near $x/|x| = -\overline{\omega}$, we must use the second ansatz, as in Section 5.1 or [18].

The subsequent errors are solved away in exactly the same manner, resulting in an approximation as claimed.

6. End of proof of structure results

In order to finish the proof of Theorem 1.1, we must tie up a number of loose ends. These include showing the existence of a function u as in the definition of the Poisson operator (Definition 4.1), proving Proposition 4.3, and showing that the construction of the (partial) Poisson operators in the previous sections captures all the singularities of the scattering matrix. As these use similar techniques, we give the proofs in this section.

For $\lambda \in \mathbb{R} \setminus \{0\}$, $\epsilon > 0$, the limit

$$\begin{aligned}\lim_{\delta\downarrow 0}\langle z\rangle^{-1/2-\epsilon}(c^2\Delta - (\lambda - i\delta)^2)^{-1}c^2\langle z\rangle^{-1/2-\epsilon} &= \lim_{\delta\downarrow 0}\langle z\rangle^{-1/2-\epsilon}(\Delta - c^{-2}(\lambda - i\delta)^2)^{-1}\langle z\rangle^{-1/2-\epsilon}\\ &= \langle z\rangle^{-1/2-\epsilon}(\Delta - c^{-2}(\lambda - i0)^2)^{-1}\langle z\rangle^{-1/2-\epsilon}\end{aligned}$$

as an operator on $L^2(\mathbb{R}^n)$ exists in the norm topology ([4, 9]). In this section, we study further properties of $(\Delta - (\lambda - i0)^2c^{-2})^{-1}$ when it is applied to a function $f \in \langle z\rangle^{-\infty}L^2(\mathbb{R}^n)$, $(1 - \phi(y))f \in \mathcal{S}(\mathbb{R}^n)$ for some $\phi \in C_c^\infty(\mathbb{R})$. In particular, we are interested in the asymptotics at infinity of the resulting function.

For simplicity of exposition, we shall assume $\lambda > 0$ throughout this section. The results for $\lambda < 0$ can be proved in a similar way.

6.1. "Outgoing" solutions.

The main result of this section is Proposition 6.1. This proposition, combined with the approximation of the Poisson operator in the previous section, shows the existence of a function u with the properties given in Definition 4.1. Taken together with Proposition 4.2, this shows the existence and uniqueness of the Poisson operator.

In proving this proposition, as well as in many others, we shall use the fact that if

$$u = \lim_{\epsilon\downarrow 0}(\Delta - c^{-2}(\lambda - i\epsilon)^2)^{-1}f = (\Delta - c^{-2}(\lambda - i0)^2)^{-1}f,$$

then

$$u = (\Delta - (\lambda - i0)^2c_0^{-2})^{-1}(Vu + f) \tag{25}$$

where

$$V = \lambda^2(c^{-2} - c_0^{-2}). \tag{26}$$

Additionally,

$$(\Delta - (\lambda - i0)^2c_0^{-2})^{-1}g(z) = (2\pi)^{1-n}\int e^{ix\cdot\xi}((D_y^2 + |\xi|^2 - (\lambda - i0)^2c_0^{-2})^{-1}\hat{g}(\xi, \cdot))(y)d\xi \tag{27}$$

where $\hat{g}(\xi, y) = \int e^{-ix\cdot\xi} g(x,y)dx$ is the Fourier transform in the x variables only. We shall repeatedly use this notation for the Fourier transform in the x variables only, and V is given by (26).

We make several remarks about the operator $(D_y^2 + t^2 - (\lambda - i0)^2 c_0^{-2})^{-1}$. As an operator from $(D_y^2 + t^2 - (\lambda - i0)^2 c_0^{-2})^{-1} : \langle y\rangle^{-1/2-\epsilon} L^2(\mathbb{R}) \to \langle y\rangle^{1/2+\epsilon} L^2(\mathbb{R})$ for any $\epsilon > 0$ it is smooth for $|t| < \lambda/c_+$, t away from $\lambda/c_\pm$. Near $t = \lambda/c_+$, $(\lambda^2/c_+^2 - t^2)^{1/2}(D_y^2 + t^2 - (\lambda - i0)^2 c_0^{-2})^{-1}$ is a smooth function of $(\lambda^2/c_+^2 - t^2)^{1/2}$, and, if $c_- > c_+$, near $t = \lambda/c_-$ it is a smooth function of $(\lambda^2/c_-^2 - t^2)^{1/2}$.

We shall use the following lemma in the proof of Proposition 6.1.

Lemma 6.1. *If $u = \lim_{\epsilon\downarrow 0}(\Delta - (\lambda - i\epsilon)^2 c^{-2})^{-1} f$ and $f \in \langle z\rangle^{-\infty} L^2$, then $\widehat{Vu}(\xi, y)$ and all its derivatives with respect to $\xi/|\xi|$ are, for $\epsilon > 0$, in $H^{J-1/2-\epsilon-\beta}(\mathbb{R}^{n-1}_\xi; \langle y\rangle^{-\beta} L^2(\mathbb{R}_y))$ for $0 \leq \beta \leq J - 1/2 - \epsilon$.*

Proof. Throughout the proof $\epsilon > 0$ is small, and may change from line to line.

We have $(\Delta - \lambda^2/c_0^2)u = Vu + f$. Then

$$\widehat{Vu}(\xi, y) = C\int e^{-ix\cdot\xi} V(x,y) \int e^{ix\cdot\eta}((D_y^2 + |\eta|^2 - c_0^2(\lambda - i0)^2)^{-1}[\hat{f}(\eta,\cdot) + \widehat{Vu}(\eta,\cdot)])(y)d\eta dx. \tag{28}$$

Let $D_{(\xi/|\xi|)_l}$ stand for a derivative tangent to $|\xi|$ =constant. Then

$$\begin{aligned}
&\left(D_{(\xi/|\xi|)_l}\right)^j \int e^{-ix\cdot\xi} V(x,y) \int e^{ix\cdot\eta}((D_y^2 + |\eta|^2 - c_0^{-2}(\lambda - i0)^2)^{-1}\widehat{Vu}(\eta,\cdot))(y)d\eta dx \\
&\quad = \int e^{-ix\cdot\xi} V(x,y) \int e^{ix\cdot\eta}((D_y^2 + |\eta|^2 - c_0^{-2}(\lambda - i0)^2)^{-1} \left(D_{(\eta/|\eta|)_l}\right)^j \widehat{Vu}(\eta,\cdot))(y)d\eta dx \\
&\quad + \sum_{|\alpha|<j} \int e^{-ix\cdot\xi} V_\alpha(x,y,\xi) \int e^{ix\cdot\eta}((D_y^2 + |\eta|^2 - c_0^{-2}(\lambda - i0)^2)^{-1} \left(D_{(\eta/|\eta|)}\right)^\alpha \widehat{Vu}(\eta,\cdot))(y)d\eta dx.
\end{aligned} \tag{29}$$

Here $D^\alpha_{(\eta/|\eta|)}$ is a derivative of order $|\alpha|$ in the $\xi/|\xi|$ variables and

$$|D_x^\beta D_y^k D_\xi^\gamma V_\alpha(x,y,\xi)| \leq C(1 + |z|)^{-J-|\beta|-k} \tag{30}$$

for large $|z|$, where if $k > 0$, we need also require $|y| > y_M$. That is, V_α is a symbol in x, a property we shall use. Let $\Delta_{\xi/|\xi|}$ be the Laplacian in the $\xi/|\xi|$ variables. If $0 < t < 1/2$, then

$$\begin{aligned}
&\left(\Delta_{\xi/|\xi|}\right)^t \int e^{-ix\cdot\xi} V(x,y) \int e^{ix\cdot\eta}((D_y^2 + |\eta|^2 - c_0^{-2}\lambda^2)^{-1}\widehat{Vu}(\eta,\cdot))(y)d\eta dx \\
&\quad = \int e^{-ix\cdot\xi} V(x,y) \int e^{ix\cdot\eta}((D_y^2 + |\eta|^2 - c_0^{-2}\lambda^2)^{-1} \left(\Delta_{\eta/|\eta|}\right)^t \widehat{Vu}(\eta,\cdot))(y)d\eta dx \\
&\quad + \int e^{-ix\cdot\xi} V_t(x,y,\xi) \int e^{ix\cdot\eta}((D_y^2 + |\eta|^2 - c_0^{-2}\lambda^2)^{-1}\widehat{Vu}(\eta,\cdot))(y)d\eta dx
\end{aligned} \tag{31}$$

where $V_t(x,y,\xi)$ satisfies, for large $|z|$,

$$|D_x^\alpha D_y^m D_\xi^\gamma V_t(x,y,\xi)| \leq C_{\alpha,t,\gamma,m}(1 + |z|)^{-J-1+2t-|\alpha|-m}$$

where we require $m = 0$ if $|y| \leq y_M$.

Because of the decay properties of V, $\widehat{Vu}(\eta, y) \in H^{J-1/2-\epsilon-\beta}(\mathbb{R}^{n-1}_\eta; \langle y\rangle^{-\beta} L^2(\mathbb{R}_y))$ for $J - \epsilon - 1/2 \geq \beta \geq 0$. Of course, $\hat{f}(\eta, y) \in H^\infty(\mathbb{R}^{n-1}_\eta; \langle y\rangle^{-\infty} L^2_y)$.

Suppose for some $t > 0$, $(\Delta_{\xi/|\xi|})^{t/2}\widehat{Vu} \in H^{J-1/2-\epsilon-\beta}(\mathbb{R}^{n-1}_\xi; \langle y\rangle^{-\beta} L^2(\mathbb{R}_y))$ for $J - \epsilon - 1/2 \geq \beta \geq 0$. Then using equations (28), (29) and (31) we obtain $(\Delta_{\xi/|\xi|})^{(t+2J-2-2\epsilon)/2}\widehat{Vu} \in H^{J-1/2-\epsilon-\beta}(\mathbb{R}^{n-1}_\xi; \langle y\rangle^{-\beta} L^2(\mathbb{R}_y))$. Since $J > 1$, the proof follows by induction. $\square$

Lemma 6.2. *If $w, \eta \in \mathbb{R}^m$, and*

$$|D_z^\alpha D_\eta^\beta a(w,\eta)| \leq C_{\alpha\beta}(1+|w|)^{-|\beta|}$$

for all multiindices α, β, then the operator A defined by

$$(Ag)(\eta) = \int e^{iw\cdot\eta} a(w,\eta) g(w) dw$$

is a continuous map

$$A : \langle w\rangle^\gamma L^2(\mathbb{R}^m) \to H^{-\gamma}(\mathbb{R}^m).$$

Proof. We can write

$$(Ag)(\eta) = \int e^{iw\cdot(\eta-\eta')} a(w,\eta)\check{g}(\eta') d\eta' dw$$

with $\check{g} \in H^{-\gamma}(\mathbb{R}^m)$ the inverse Fourier transform on $\mathbb{R}^m$. Since a is a symbol of order 0 in w, this is a pseudodifferential operator acting on $\check{g}$ and the result follows from standard pseudodifferential operator theory. $\square$

Lemma 6.3. *Let $a(w,\eta)$ be a symbol of order 0 in $w = (\overline{w}, w_{m+1}) \in \mathbb{R}^{m+1}$, and let $h(\eta) \in H^{-\gamma}(\mathbb{R}^m)$ be supported away from $t^2 = |\eta|^2$, where t is a nonzero constant. Then*

$$(Ah)(w) = \int e^{i\overline{w}\cdot\eta + iw_{m+1}\sqrt{t^2-|\eta|^2}} a(w,\eta) h(\eta) d\eta \in \langle w\rangle^{\gamma+1/2} L^2(\mathbb{R}^{m+1})$$

provided $\gamma > 0$.

Proof. Let $g \in \langle w\rangle^{-\gamma-1/2} L^2(\mathbb{R}^{m+1})$. Then

$$(g, Ah) = (\int \overline{a}(w,\eta) e^{-i\overline{w}\cdot\eta - iw_{m+1}\tau} g(w) dw_{|\tau=\sqrt{t^2-|\eta|^2}}, h(\eta)). \tag{32}$$

By the previous lemma and the restriction properties of elements of Sobolev spaces,

$$\int \overline{a}(w,\eta) e^{-i\overline{w}\cdot\eta - iw_{m+1}\tau} g(w) dw_{|\tau=\sqrt{t^2-|\eta|^2}} \in H^\gamma$$

if $\gamma > 0$. The pairing (32) is then well-defined for all such g, and $Ah \in \langle w\rangle^{\gamma+1/2} L^2(\mathbb{R}^{m+1})$. $\square$

Proposition 6.1. *If $f \in \langle z\rangle^{-\infty} L^2(\mathbb{R}^n)$ and $J \geq 2$, then $u = (\Delta - (\lambda - i0)^2 c^{-2})^{-1} f$ is outgoing in the sense of Definition 4.2.*

Proof. We use (25) and (27). Choose $\Psi \in C_c^\infty(\mathbb{R})$ to be 1 for $|\xi| \leq \lambda^2/c_+^2$ and supported in a slightly larger neighborhood. We use the fact that we can write, for $\delta > 0$, $(1-\Psi(\xi))((D_y^2 + |\xi|^2 - (\lambda - i\delta)^2 c_0^{-2})^{-1}$ as a sum of an operator bounded on $L^2(\mathbb{R})$ and an operator involving projection onto the discrete spectrum. The part corresponding to an operator bounded on L^2 gives an element of L^2. For the part corresponding to the discrete spectrum, near the poles we move the contour of integration in $|\xi|$ slightly into the upper half-plane when $x\cdot\xi \geq 0$ and into the lower half-plane for $x \cdot \xi \leq 0$, obtaining

$$\int e^{ix\cdot\xi}(1-\Psi(|\xi|))((D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1}(\widehat{Vu} + \widehat{f})(\xi,\cdot))(y) d\xi = \sum_1^{T(\lambda)} e^{-i\kappa_j(\lambda)|x|} b_j(\frac{x}{|x|}) f_j(y) + \tilde{u}_1$$

where $\tilde{u_1}, \frac{\partial}{\partial|z|}\tilde{u}_1 \in L^2(\mathbb{R}^n)$. We remark here that $b_j \in C^\infty(\mathbb{S}^{n-1})$, thanks to the fact that it originates from a stationary phase applied to functions smooth in $\xi/|\xi|$ (Lemma 6.1) at $|\xi| = \kappa_j(\lambda)$. We refer the reader to the proof of Theorem 4.1 of [5] for greater detail in a very similar computation.

When $|y| < C$ for some constant C, we have that away from $|\xi| = \lambda/c_+$,

$$w(\xi, y) = \Psi(|\xi|)((D_y^2 + |\xi|^2 - (\lambda - i0)^2)^{-1}(\widehat{Vu} + \widehat{f})(\xi,\cdot))(y) \in L^2(\mathbb{R}_\xi^{n-1})$$

with a norm independent of y, $|y| < C$. Near $|\xi| = \lambda/c_+$, $w(\xi, y) \in L^p(\mathbb{R}^{n-1})$ for any $p < 2$, again with norm independent of y, $|y| < C$, so using the mapping properties of the Fourier transform we have

$$\int e^{ix\cdot\xi} w(\xi, y) d\xi_{|\{|y|<C\}} \in \langle x\rangle^{\epsilon} L^2(\mathbb{R}^{n-1}_x \times [-C, C]_y).$$

When $y > y_M$, for $|\xi| \leq \lambda/c_+$ we have

$$((D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1}\widehat{Vu} + \widehat{c^2 f}(\xi, \cdot))(y) = h_1(y) e^{-i\sqrt{\lambda^2/c_+^2 - |\xi|^2}} + h_2(y, \xi).$$

Since $h_2(y, \xi) \in L^2(\mathbb{R}^{n-1}_\xi; \langle y\rangle^{-J+3/2-\epsilon} L^2(\mathbb{R}_y))$, taking its inverse Fourier transform in x gives an element of $\langle y\rangle^{-J+3/2-\epsilon} L^2(\mathbb{R}^n \cap \{y > y_M\})$, and similarly if we take a radial derivative.

For the term $h_1(y) e^{-i\sqrt{\lambda^2/c_+^2 - |\xi|^2}}$, note that away from $|\xi| = \lambda/c_\pm$, $h_1 \in H^{J-1-\epsilon}(\mathbb{R}^{n-1})$. Notice that

$$\left(\frac{\partial}{\partial|z|} + i\lambda/c_+\right) e^{ix\cdot\xi - iy\sqrt{\lambda^2/c_+^2 - |\xi|^2}} = \left(\frac{x}{|z|}\cdot\xi - i\frac{y}{|z|}\sqrt{\lambda^2/c_+^2 - |\xi|^2} + i\lambda/c_+\right) e^{ix\cdot\xi - iy\sqrt{\lambda^2/c_+^2 - |\xi|^2}}, \tag{33}$$

that is, it vanishes to first order on the critical points of the phase. Therefore, smoothly restricting our region of integration to $|\xi| < \lambda/c_+ - \delta$ and away from $|\xi| = \lambda/c_-$, an integration by parts argument shows us that we get an element of $\langle z\rangle^{\epsilon} L^2(\mathbb{R}^n \cap \{y > y_M\})$ for any $\epsilon > 0$.

Near $|\xi| = \lambda/c_+$, $|\xi| \leq \lambda/c_+$, $h_1 = (\lambda^2/c_+^2 - |\xi|^2)^{-1/2} b(\sqrt{\lambda^2/c_+^2 - |\xi|^2}) h_+(\xi)$, where $b \in C^\infty$ and $h_+(\xi) = h_{++}(\xi, \sqrt{\lambda^2/c_+^2 - |\xi|^2})$, $h_{++} \in H^{J-1/2-\epsilon}(\mathbb{R}^n)$. Using a partition of unity, we can work on coordinate patches on which $\xi_i \neq 0$ for some i. For example, let $\overline{\xi} = (\xi_2, \xi_2, ...\xi_{n-1})$ and for $\xi_1 > \delta > 0$, we use $(\overline{\xi}, t = \sqrt{\lambda^2/c_+^2 - |\xi|^2})$ as our variables of integration. Then we need to compute

$$\left(\frac{\partial}{\partial|z|} + i\lambda/c_+\right) \int_{t\geq 0} e^{i(x_1\sqrt{\lambda^2/c_+^2 - t^2 - |\overline{\xi}|^2} + \overline{x}\cdot\overline{\xi} - yt)} \chi_1(\overline{\xi}, t) b(t) h_{++}(\sqrt{\lambda^2/c_+^2 - t^2 - |\overline{\xi}|^2}, \overline{\xi}, t) dt.$$

Here χ_1 is a smooth cut-off function and $\overline{x} = (x_2, x_3, ..., x_{n-1})$. The resulting integrand vanishes at the stationary points of the phase, so that we may integrate by parts. Applying Lemma 6.3 and a related result for the boundary term, we obtain an element of $\langle z\rangle^{\epsilon} L^2(\mathbb{R}^n)$. We can repeat this procedure, using the partition of unity, to cover the integration over the region with $|\xi| \leq \lambda/c_+$, $|\xi|$ near λ/c_+.

For $|\xi| \geq \lambda/c_+$, we have

$$\int_{|\xi|\geq\lambda/c_+} \Psi(|\xi|) e^{ix\cdot\xi} (|\xi|^2 - \lambda^2/c_+^2)^{-1/2} \Big(e^{-y\sqrt{|\xi|^2 - \lambda^2/c_+^2}} \int_{-\infty}^{y} a(\sqrt{|\xi|^2 - \lambda^2/c_+^2}) (\widehat{Vu} + \hat{f})(\xi, y') \phi(y', \sqrt{|\xi|^2 - \lambda^2/c_+^2}) dy' + h_{2b}(\xi, y) \Big) d\xi \tag{34}$$

with $a(t)$, $\phi(y', t)$ smooth in t and $|\phi(y', t)| \leq Ce^{y't}$. The function $\phi(y, \sqrt{|\xi|^2 - \lambda^2/c_+^2})$ is in the null space of $D_y^2 + |\xi|^2 - \lambda^2/c_0^2$. Since $h_{2b}, D_y h_{2b} \in H^{J-1-\epsilon-\gamma}(\mathbb{R}^{n-1}; \langle y\rangle^{-\gamma} L^2(\mathbb{R}))$ for $J - 1 - \epsilon - \gamma \geq 0$, its contribution is in $\langle z\rangle^{\epsilon} L^2(\mathbb{R}^n)$ and can be neglected. As in the previous integral, for the rest we use a partition of unity to work in sets with $\xi_i \neq 0$, some i. Near $\xi_1 \neq 0$, we use coordinates $\overline{\xi}, t = \sqrt{|\xi|^2 - \lambda^2/c_+^2}$ almost as before. Using the technique of the proof of Lemma 6.3, applying $(\frac{\partial}{\partial|z|} + i\lambda/c_+)$ when $y > y_M$, integrating by parts, and using Lemma 6.2, we need to check only that

$$\int \int_{t\geq 0} \int_{y_M}^{\infty} w_1(\overline{\xi}, t, y) \int_{y_M}^{y} e^{-yt} \phi(y', t) w_2(\overline{\xi}, t, y') dy' dt d\overline{\xi} \tag{35}$$

is finite when $w_1 \in H^{\epsilon/2}(\mathbb{R}^{n-1}; \langle y\rangle^{-1-\epsilon/2} L^2(\mathbb{R}))$ and $w_2 \in H^{J-1-2\delta}(\mathbb{R}^{n-1}; \langle y\rangle^{-\delta} L^2(\mathbb{R}))$ for any $\delta > 0$, where w_2 is supported near $t = 0$ and away from $|\overline{\xi}| = \lambda/c_+$. Since by Hölder's inequality

$$|\int_{y_M}^{y} e^{-(y-y')t} w_2(\overline{\xi}, t, y') dy'| \leq C\langle y\rangle^{-\delta+1/2} (\int |w_2(\overline{\xi}, t, y')|^2 \langle y'\rangle^{2\delta} dy')^{1/2}$$

for any $\delta > 0$, using Hölder's inequality again we see that (35) is finite.

Near $|\xi| = \lambda/c_-$, $h_1 = h_{-1} + h_{-2}$, where $h_{-2} \in H^{J-1-\epsilon}$ and can be treated by an integration by parts as before. However, h_{-1} is the restriction of an element of $H^{J-1/2-\epsilon}$ to the hypersurface $(\xi, \sqrt{\lambda^2/c_-^2 - |\xi|^2})$. Here we introduce the variable of integration $t = \sqrt{\lambda^2/c_-^2 - |\xi|^2}$; this results in an extra factor of t in the integrand and makes $t = 0$ a stationary point in of the phase. However, the extra factor of t in the integrand is enough to allow us to integrate by parts after applying $\frac{\partial}{\partial |z|} + i\lambda/c_+$, and arguing as in Lemma 6.3, we obtain an element of $\langle z\rangle^{\epsilon} L^2(\mathbb{R}^n \cap \{y > y_M\})$.

A similar technique shows that $(\frac{\partial}{\partial |z|} + i\lambda/c_-)(\Delta - (\lambda - i0)^2 c^{-2})^{-1} f_{|y<-y_M} \in \langle z\rangle^{\epsilon} L^2(\mathbb{R}^n \cap \{y < -y_M\})$ for any $\epsilon > 0$. □

For future reference, we remark that the proof above has shown

Corollary 6.1. *If $f \in \langle z\rangle^{-\infty}(\mathbb{R}^n)$, then for $y \in K$, $K \subset \mathbb{R}$ compact, $|x| > 1$,*

$$(\Delta - c^{-2}(\lambda - i0)^2)^{-1} f(z)_{||x|>1, y\in K} = |x|^{-(n-2)/2} \sum_{1}^{T(\lambda)} e^{-i\kappa_j(\lambda)|x|} b_j(x/|x|) f_j(y) + u_1$$

where $b_j \in C^\infty(\mathbb{S}^{n-2})$ and $u_1 \in \langle x\rangle^{\epsilon} L^2(\mathbb{R}^{n-1} \times K)$.

6.2. **Smoothness of leading order coefficient.** To finish the proof of Theorem 1.1, we need to prove Theorem 1.3, which we recall:

Theorem . *Let c and c_0 satisfy the hypotheses of Section 2 and hypothesis (H1) or (H2). For any $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$, $f \in \langle z\rangle^{-\infty} L^2(\mathbb{R}^n)$, $(1 - \phi(y))f \in \mathcal{S}(\mathbb{R}^n)$ for some $\phi \in C_c^\infty(\mathbb{R})$, we have*

$$\chi(z/|z|)(\Delta - c^{-2}(\lambda - i0)^2)^{-1} f = e^{-i\lambda|z|/c} |z|^{-(n-1)/2} a_0(z/|z|) + u_1$$

where $a_0 \in C^\infty(\mathbb{S}_c^{n-1})$ and $u_1 \in \langle z\rangle^{\epsilon} L^2(\mathbb{R}^n)$ for all $\epsilon > 0$.

The proof of this theorem will occupy the next two subsections. Before we prove it, however, we give some applications.

Proof of Proposition 4.3. We have $P(\lambda)\Pi_j = \tilde{P}_j(\lambda) - (\Delta - c^{-2}(\lambda - i0)^2)^{-1}(\Delta - \lambda^2 c^{-2})\tilde{P}_j(\lambda)$, where $\tilde{P}_j$ are the approximate (partial) Poisson operators constructed in Section 5. Proposition 4.3 now follows from our construction of $\tilde{P}_j$, Proposition 6.1, Corollary 6.1 and Theorem 1.3. □

Moreover, if we take $f = (\Delta - \lambda^2 c^{-2})\tilde{P}_0(\lambda, z, \omega)$, where $\tilde{P}_0$ is the approximation of the (partial) Poisson operator we have constructed, then this theorem shows that the the only singularities of the main part of the scattering matrix come from the structure of the approximation of the Poisson operator we have constructed. This then proves Theorem 1.1.

The following proposition implies Theorem 1.3 in one case.

Proposition 6.2. *Suppose c and c_0 satisfy the hypothesis (H1). Let $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$ and let $f \in \langle z\rangle^{-\infty} L^2$, $(1 - \phi(y))f \in \mathcal{S}(\mathbb{R}^n)$ for some $\phi \in C_c^\infty(\mathbb{R})$. Then*

$$\chi(z/|z|)(\Delta - c^{-2}(\lambda - i0)^2)^{-1} f = |z|^{-(n-1)/2} e^{-i\lambda|z|/c_0(y)} \sum_{0}^{N} |z|^{-j} a_j(z/|z|) + \mathcal{O}(|z|^{-(n-1)/2-N-1})$$

where $a_j \in C^\infty(\mathbb{S}_c^{n-1})$.

Proof. Because $c_+ = c_-$, we can recast this in the form of a perturbation of a simple n-body problem and use the fact that much is known about asymptotic expansions of the resolvent applied to Schwartz function. First, we note that

$$(\Delta - c^{-2}(\lambda - i0)^2)^{-1} = (\Delta - \lambda^2(c^{-2} - c_+^{-2}) - c_+^{-2}(\lambda - i0)^2)^{-1}.$$

The operator $\Delta - \lambda^2(c_0^{-2} - c_+^{-2})$ is a particularly simple example of a class of n-body operators widely studied. The operator $\Delta - \lambda^2(c^{-2} - c_+^{-2})$, while not quite an n-body operator since the potential depends on all variables, is a perturbation that has many of the same properties we desire.

The paper [24], which builds on results of [13, 22], shows that

$$\chi(z/|z|)(\Delta - \lambda^2(c_0^{-2} - c_+^{-2}) - c_+^{-2}(\lambda - i0)^2)^{-1} f$$
$$= |z|^{-(n-1)/2} e^{-i\lambda|z|/c_0(y)} \sum_0^N |z|^{-j} b_j(z/|z|) + \mathcal{O}(|z|^{-(n-1)/2-N-1})$$

with b_j smooth. The proof is such that the results of [24] hold with c_0 replaced by a sound speed c of the type considered here. Roughly speaking, this is because [13] requires that the operator $\Delta + V_1$ (where for us $V_1 = -\lambda^2(c_0^{-2} - c_+^{-2})$ or $V_1 = -\lambda^2(c^{-2} - c_+^{-2})$) satisfy a Mourre estimate and some regularity and decay properties, both of which are satisfied for either V_1. Vasy remarks already that the results of [24, Section 2], which are local versions of results of [22], will hold in our case. Then the results of [24, Section 3] hold for our case, since just as in that paper we can argue that from the results of [13] and [22] that

$$\mathrm{WF}^*_{\mathrm{sc}}\left(\chi(z/|z|)(\Delta - c^{-2}(\lambda - i0)^2)^{-1} f\right) \subset R^+_{\lambda/c_+},$$

and the asymptotic expansion follows from [24, Proposition 2.8] and the remarks made there. (Roughly speaking, $\mathrm{WF}_{\mathrm{sc}}(u)$ provides a microlocal description of the lack of decay of u– see [22] for a definition of the scattering wave front set, $\mathrm{WF}_{\mathrm{sc}}$, and R^+_τ.) □

Before giving the proof in the case $c_+ \neq c_-$, we give some explanation as to why the proof of [24] does not apply in this case. The argument of [24] uses very strongly the fact that, for $f \in \mathcal{S}$,

$$\mathrm{WF}^*_{\mathrm{sc}}\left(\chi(z/|z|)(\Delta - \lambda^2(c_0^{-2} - c_+^{-2}) - c_+^{-2}(\lambda - i0)^2)^{-1} f\right) \subset R^+_{\lambda/c_+}$$

where $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$.

This is not, however, true in general when $c_+ \neq c_-$. The results of [5, Theorem 4.1] show that in general for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\left((\Delta - (\lambda - i0)^2 c_0^{-2})^{-1} f\right)(z) = e^{-i\lambda|z|/c_+}|z|^{-(n-1)/2} a_0(z/|z|)$$
$$+ |z|^{-(n+1)/2}\left(e^{-i\lambda|z|/c_+} a_1(z/|z|) + e^{-i\lambda|x|/c_- - iy\lambda\sqrt{1/c_+^2 - 1/c_-^2}} b_1(z/|z|)\right) + \mathcal{O}(|z|^{-(n+3)/2}) \quad (36)$$

when $0 < \epsilon < y/|z| < (1 - c_+^2/c_-^2)^{1/2} - \epsilon$ and $|z| \to \infty$. If $b_1 \neq 0$, then the scattering wave front set is not contained in R^+_{λ/c_+}. This means that the scattering wave front set of $\chi(\Delta - (\lambda - i0)^2/c^2)^{-1} f$ is in general more complicated than the $c_+ = c_-$ case (and unknown, to the best of our knowledge), and the techniques of [22, 24] cannot be immediately applied. Similar differences can be seen in the resolvent estimates of [13, Theorem 1.1] for the n-body problem, and [16, Theorem 3.1], for a particular stratified medium.

Instead, we take a different approach.

6.3. **Proof of Theorem 1.3 in case hypothesis (H2) holds.** In order to prove Theorem 1.3 when hypothesis (H2) holds, we will make heavy use of equations (25) and (27). We use the fact that the more rapidly g decays at infinity, the more we can say about $(\Delta - (\lambda - i0)^2 c_0^{-2})^{-1} g$, using equation (27). To take advantage of this, roughly speaking, we find approximations w of $u = (\Delta - (\lambda - i0)^2 c^{-2})^{-1} f$ so that $(\Delta - \lambda^2/c_0^2) w = Vu + e$, with the error e decaying faster than Vu does, and w decaying faster than u. Then

$u = w - (\Delta - (\lambda - i0)^2 c_0^{-2})^{-2}(f + e)$ (compare (25)). The better rate of decay of e improves our knowledge of u.

In practice, the proof is somewhat more complicated. We study the behaviour at infinity of $\chi(z/|z|)u$, $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$, and we introduce a "microlocal" cut-off $\Psi(D_x)$ so that $\chi\Psi(D_x)u$ has the same leading behaviour as χu at infinity (Lemma 6.4), but is easier to understand.

Lemma 6.4 shows that, for suitable Ψ, $\chi(1 - \Psi(D_x))u \in \langle z\rangle^{\epsilon-1/2}L^2(\mathbb{R}^n)$, so that $\chi\Psi(D_x)u$ captures the leading behaviour of χu. Lemmas 6.5-6.9 are preliminaries, and Lemmas 6.10-6.11 are used to make successive approximations of χu in the proof of Proposition 6.3, which shows that $\chi\Psi(D_x)u$ has an asymptotic expansion with smooth coefficient in the leading order term.

The next lemma shows us that, for suitable Ψ and χ, $\chi\Psi(D_x)u$ is the leading order term of χu.

Lemma 6.4. *Let $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$, $\Psi \in C_b^\infty(\mathbb{R}^{n-1})$, and suppose that if $(x/|z|, y/|z|) \in \operatorname{supp}\chi$ and $y > 0$, then $\pm\lambda x/(c_+|z|) \notin \operatorname{supp}(1 - \Psi)$, and if $(x/|z|, y/|z|) \in \operatorname{supp}\chi$ and $y < 0$, then $\pm\lambda x/(c_-|z|) \notin \operatorname{supp}(1 - \Psi)$. If $J \geq 4$, $u = (\Delta - (\lambda - i0)^2 c^{-2})^{-1} f$, and $f \in \langle z\rangle^{-\infty}L^2(\mathbb{R}^n)$, then*

$$\chi(z/|z|)(1 - \Psi(D_x))u,\ \frac{\partial}{\partial|z|}\chi(z/|z|)(1 - \Psi(D_x))u \in \langle z\rangle^{-1/2+\epsilon}L^2(\mathbb{R}^n)$$

for any $\epsilon > 0$.

Proof. We give the proof for χ supported in $y > 0$, as the proof for χ with support in $y < 0$ is quite similar.

The proof closely resembles that of Proposition 6.1. Let $\Psi_1 \in C_b^\infty(\mathbb{R})$ be such that $\operatorname{supp}\Psi_1(t) \subset \{|t| > \lambda/c_+\}$, $\operatorname{supp}(1 - \Psi_1) \subset \{|t| \leq \lambda/c_+ + \delta\}$, some small $\delta > 0$. Then, by the same type of arguments as in the proof of Proposition 6.1, since the eigenfunctions of $D_y^2 + c_0^{-2}\lambda^2$ are exponentially decreasing in y, we have

$$\chi(z/|z|)\int e^{ix\cdot\xi}(1 - \Psi(\xi))\Psi_1(|\xi|)(D_y^2 + |\xi| - c_0^{-2}(\lambda - i0)^2)^{-1}(\widehat{Vu}(\xi, \cdot) + \hat{f}(\xi, \cdot))(y)d\xi \in \langle z\rangle^{-1/2+\epsilon}L^2(\mathbb{R}^n).$$

Moreover, for $y > y_M$, using (33), we see that when z is restricted to the support of $\chi(z/|z|)$ there are no stationary points of the associated phase $x \cdot \xi - y\sqrt{\lambda^2/c_+^2 - |\xi|^2}$ on the support of $1 - \Psi(\xi)$. Therefore, when the integrand is supported in $|\xi| < \lambda/c_+$ and away from $|\xi| = \lambda/c_\pm$, we can integrate by parts as in the proof of Proposition 6.1 and get an element of $\langle z\rangle^{-1/2+\epsilon}L^2(\mathbb{R}^n)$. We need only examine the integration near $|\xi| = \lambda/c_\pm$ more closely.

If $c_+ < c_-$, near $|\xi| = \lambda/c_-$, $y > y_M$, we have

$$(D_y + |\xi|^2 - (\lambda - i0)^2)^{-1}(\widehat{Vu}(\xi, \cdot) + \hat{f}(\xi, \cdot))(y) = e^{-iy\sqrt{\lambda^2/c_+^2 - |\xi|^2}}(a_1(\xi) + (\lambda^2/c_-^2 - |\xi|^2)^{1/2}a_2(\xi)) + b(y, \xi) \tag{37}$$

where $a_1, a_2 \in H^l(\mathbb{R}^{n-1})$, l an integer with $l < J - 1$, and $b \in L^2(\mathbb{R}^{n-1}; \langle y\rangle^{-J+1+\epsilon}L^2(\mathbb{R}))$. Since again there are no stationary points of the phase $x \cdot \xi - y\sqrt{\lambda^2/c_+^2 - |\xi|^2}$ with $z/|z|$ in the support of χ and ξ in the support of $1 - \Psi$, we integrate by parts. Since, for any $\Psi_3 \in C_c^\infty(\mathbb{R})$, the Fourier transform in x of $\Psi_3(|\xi| - \lambda/c_+)(\lambda^2/c_-^2 - |\xi|^2)^{-1/2}$ is in $L^p(\mathbb{R}^{n-1}) \cap C^\infty(\mathbb{R}^{n-1})$ for any $p > 2$, we have the contribution of the integration of a_i terms over this region is in $\langle z\rangle^{-1/2+\epsilon}L^2(\mathbb{R}^n)$ for any $\epsilon > 0$. Clearly, the contribution of the integration of the b from (37) is in $\langle z\rangle^{-J+1+\epsilon}L^2(\mathbb{R}^n)$.

Near $|\xi| = \lambda/c_+$, as in the proof of Proposition 6.1, we divide the integration into two pieces. When $|\xi| < \lambda/c_+$, we introduce the coordinate $t = \sqrt{\lambda^2/c_+^2 - |\xi|^2}$ as in the proof of Proposition 6.1 and can integrate by parts to get an element of $\langle z\rangle^{-1/2+\epsilon}L^2(\mathbb{R}^n)$.

For the part with $|\xi| \geq \lambda/c_+$, we use (34). Changing coordinates as in the proof of Proposition 6.1, we can, in the region with $\xi_1 > \delta > 0$, consider the phase to be $x_1\sqrt{t^2 + |\overline{\xi}|^2 - \lambda^2/c_+^2} + \overline{x} \cdot \overline{\xi} + iyt$, using

$t = \sqrt{|\xi|^2 - \lambda^2/c_+^2}$ and the notation of the proof of Proposition 6.1. Again we have no stationary points and integrating by parts gives an element of $\langle z\rangle^{-1/2+\epsilon}L^2(\mathbb{R}^n)$.

A similar argument gives the same result for $\frac{\partial}{\partial|z|}\chi(1-\Psi(D_x))u$. $\square$

We use the notation $\mathbb{S}^{n-1}_\pm = \{\omega = (\overline{\omega}, \omega_n) \in \mathbb{S}^{n-1} : \pm\omega_n > 0\}$ and will use differential operators of the following type.

Definition 6.1. *We say that a differential operator $P \in \operatorname{Diff}^l_r(\mathbb{R}^n)$ if P is a differential operator of the form*

$$\sum_\pm \sum_{|\alpha|+j\le l} b_j(\frac{1}{|z|}) a_{j,\alpha,\pm}(\frac{z}{|z|})|z|^j(\frac{\partial}{\partial|z|} + i\lambda/c_\pm)^j D^\alpha_{z/|z|}$$

where $b_j \in C^\infty([0,\infty))$, $a_{j,\alpha,\pm} \in C^\infty_c(\mathbb{S}^{n-1}_\pm)$ and $D^\alpha_{z/|z|}$ is a differential operator of order $|\alpha|$ in the $z/|z|$ variables.

We shall make use of the following lemma, whose proof follows by a straightforward computation.

Lemma 6.5. *If $P \in \operatorname{Diff}^l_r(\mathbb{R}^n)$ and $b \in C^\infty(\mathbb{S}^{n-1})$, then $[\frac{\partial}{\partial|z|}, P] \in |z|^{-1}\operatorname{Diff}^l_r(\mathbb{R}^n)$, $[\frac{\partial}{\partial z/|z|}, P] \in \operatorname{Diff}^l_r(\mathbb{R}^n)$, $[|z|^{-m}b(z/|z|), P] \in |z|^{-m}\operatorname{Diff}^{l-1}_r(\mathbb{R}^n)$, and $[\Delta, P] \in |z|^{-2}\operatorname{Diff}^{l+1}_r(\mathbb{R}^n)$.*

In order to construct the desired approximation, we shall use the following lemma. In practice, when we apply this lemma, the first term will be used in solving away the error, and the subsequent terms will be of lower order. In particular, in applications h will vanish faster than g at infinity and so will $\chi_\pm(\frac{\partial}{\partial|z|} + \frac{i\lambda}{c_\pm})g$.

Lemma 6.6. *Suppose $(\Delta - \lambda^2/c_0^2)g = h$, $P \in \operatorname{Diff}^l_r(\mathbb{R}^n)$, and $\chi_\pm, v \in C^\infty(\mathbb{S}^{n-1})$ with the support of $\chi_\pm$ contained in $\mathbb{S}^{n-1}_\pm$. Then*

$$\begin{aligned}(\Delta - \lambda^2/c_0^2)\left(|z|^{-j+1}v\chi_\pm(z/|z|)Pg\right) &= (2j-2)|z|^{-j}v\chi_\pm\frac{-i\lambda}{c_\pm}Pg + \\ (2j-2)|z|^{-j}v\chi_\pm P(\frac{\partial}{\partial|z|} + \frac{i\lambda}{c_\pm})g &+ |z|^{-j-1}v\chi_\pm(P_{l+1}g) + |z|^{-j-1}\nabla_0(v\chi_\pm)\cdot\nabla_0 Pg \\ &+ |z|^{-j+1}v\chi_\pm Ph + \Delta(|z|^{-j+1}v\chi_\pm)Pg \\ &= (2j-2)|z|^{-j}v\chi_\pm\frac{-i\lambda}{c_\pm}Pg + |z|^{-j-1}P'_{l+1}g + |z|^{-j+1}v\chi_\pm Ph \end{aligned} \tag{38}$$

with $P_{l+1}, P'_{l+1} \in \operatorname{Diff}^{l+1}_r(\mathbb{R}^n)$. Here ∇_0 stands for the gradient on $\mathbb{S}^{n-1}$.

Proof. The proof follows from a straight-forward computation, using Lemma 6.5. $\square$

If $\Psi(\xi) \in C^\infty(\mathbb{R}^{n-1})$, we use the notation

$$(\Psi(D_x)f)(z) = (2\pi)^{1-n}\int e^{i(x-x')\cdot\xi}\Psi(\xi)f(x',y)dx'd\xi.$$

Note that $[\Psi(D_x), \Delta] = 0$ and $[\Psi(D_x), c_0(y)] = 0$.

Lemma 6.7. *If $\chi_1,\ \chi_2 \in C^\infty_c(\mathbb{S}^{n-1})$ with $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$, $\Psi \in \mathcal{S}(\mathbb{R}^{n-1})$, and*

$$Au = \chi_1(z/|z|)\Psi(D_x)\chi_2(z/|z|)u,$$

then $A : \langle z\rangle^\alpha L^2(\mathbb{R}^n) \to \langle z\rangle^{-\infty}L^2(\mathbb{R}^n)$ for any $\alpha \in \mathbb{R}$.

Proof. The crucial observation is that if $(x,y)/|(x,y)| \in \operatorname{supp}\chi_1$ and $(x',y)/|(x',y)| \in \operatorname{supp}\chi_2$, then $|x-x'| \ge \beta|(x,y)|$, $|x-x'| \ge \beta|(x',y)|$ for some $\beta > 0$.

The Schwartz kernel of $\Psi(D_x)$ is given by $(2\pi)^{1-n}\hat{\Psi}(x'-x)$, where $\hat{\Psi}\in\mathcal{S}(\mathbb{R}^{n-1})$. Thus, for $f\in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
|Af|(z) &= (2\pi)^{1-n}|\chi_1(z/|z|)\int\hat{\Psi}(x'-x)\chi_2((x',y)/|(x',y)|)f(x',y)dx'| \\
&\le C|\chi_1(z/|z|)\int(1+|x-x'|)^{-m-\alpha}(1+|x-x'|)^{m+\alpha}\hat{\Psi}(x'-x)\chi_2((x',y)/|(x',y)|)f(x',y)dx'| \\
&\le C\langle z\rangle^{-m}(\int(1+|x-x'|)^{2m+2\alpha}|\hat{\Psi}(x'-x)|^2dx')^{1/2}(\int\langle(x',y)\rangle^{-2\alpha}|f(x',y)|^2dx')^{1/2} \\
&\le C\langle z\rangle^{-m}(\int\langle(x',y)\rangle^{-2\alpha}|f(x',y)|^2dx')^{1/2}
\end{aligned}$$

for any m (where the constant depends on m and α) and thus it follows that $Af\in\langle z\rangle^{-\infty}L^2(\mathbb{R}^n)$. $\square$

Lemma 6.8. *If $\Psi\in C_c^\infty(\mathbb{R}^{n-1})$ and $g\in\langle z\rangle^{-\beta}L^2(\mathbb{R}^n)$, then $\Psi(D_x)g\in\langle z\rangle^{-\beta}L^2(\mathbb{R}^n)$. Suppose $D_z^\alpha g$, $D_z^\alpha Pg\in\langle z\rangle^{-\beta}L^2(\mathbb{R}^n)$ for all $P\in\mathrm{Diff}_r^k(\mathbb{R}^n)$ and for all multiindices α. Then $D_z^\alpha P\Psi(D_x)g\in\langle z\rangle^{-\beta}L^2(\mathbb{R}^n)$ for all $P\in\mathrm{Diff}_r^k(\mathbb{R}^n)$ and all multiindices α.*

Proof. To show that if $g\in\langle z\rangle^{-\beta}L^2(\mathbb{R}^n)$, so is $\Psi(D_x)g$, we take the Fourier transform of $\Psi(D_x)g$:

$$\mathcal{F}(\Psi(D_x)g)(\eta,\tau)=\Psi(\eta)\mathcal{F}(g)(\eta,\tau)$$

where here we are using $\mathcal{F}(h)(\eta,\tau)$ to denote the Fourier transform of h in all variables. Then, since $\mathcal{F}(g)\in H^\beta(\mathbb{R}^n)$, $\mathcal{F}(\Psi(D_x)g)\in H^\beta(\mathbb{R}^n)$, and $\Psi(D_x)g\in\langle z\rangle^{-\beta}L^2(\mathbb{R}^n)$.

We give an indication of the proof of the remainder of the lemma. Suppose $D_z^\alpha g$, $D_z^\alpha Pg\in\langle z\rangle^{-\beta}L^2(\mathbb{R}^n)$ for all $P\in\mathrm{Diff}_r^k(\mathbb{R}^n)$ and for all multiindices α, and $k\ge 1$. For $\chi_+\in C_c^\infty(\mathbb{S}_+^{n-1})$, consider

$$\begin{aligned}
(2\pi)^{n-1}\chi_+(z/|z|)\left(|z|\frac{\partial}{\partial|z|}+\frac{i\lambda|z|}{c_+}\right)\Psi(D_x)g &= \chi_+(z/|z|)\int\hat{\Psi}(x'-x)(x'\cdot\nabla_{x'}+y\frac{\partial}{\partial y}+i\lambda|(x',y)|/c_+)f(x',y)dx' \\
&\quad+i\lambda/c_+\chi_+(z/|z|)\int\hat{\Psi}(x'-x)(|(x,y)|-|(x',y)|)f(x'.y)dx' \\
&\quad-\chi_+(z/|z|)\int\sum\widehat{D_{\xi_j}\Psi}(x'-x)\frac{\partial}{\partial x'_j}f(x',y)dx'
\end{aligned}$$

Let $\tilde{\chi}\in C_c^\infty(\mathbb{S}_+^{n-1})$ be 1 on the support of χ. Then, using the first part of the lemma,

$$\chi_+(z/|z|)\int\hat{\Psi}(x'-x)\tilde{\chi}((x',y)/|(x',y)|)(x'\cdot\nabla_{x'}+y\frac{\partial}{\partial y}+i\lambda|(x',y)|/c_+)f(x',y)dx\in\langle z\rangle^{-\beta}L^2(\mathbb{R}^n)$$

since $(x'\cdot\nabla_{x'}+y\frac{\partial}{\partial y}+i\lambda|(x',y)|/c_+)f(x',y)\in\langle z\rangle^{-\beta}L^2(\mathbb{R}^n)$. By the same reasoning,

$$\chi_+(z/|z|)\int\sum\widehat{D_{\xi_j}\Psi}(x'-x)\frac{\partial}{\partial x'_j}f(x',y)dx'\in\langle z\rangle^{-\beta}L^2(\mathbb{R}^n).$$

Moreover, by Lemma 6.7,

$$\chi_+(z/|z|)\int\hat{\Psi}(x'-x)(1-\tilde{\chi}((x',y)/|(x',y)|))(x'\cdot\nabla_{x'}+y\frac{\partial}{\partial y}+i\lambda|(x',y)|/c_+)f(x',y)dx \\ \in\langle z\rangle^{-\infty}L^2(\mathbb{R}^n).$$

To finish, note that $\langle x-x'\rangle^m \hat{\Psi}(x'-x)(|(x,y)|-|(x',y)|)$ is a smooth, bounded, function of x, y, and x' for any m. Then, if $\beta \geq 0$,

$$\begin{aligned}
&\langle z\rangle^{2\beta} \Big| \int \hat{\Psi}(x'-x)(|(x,y)|-|(x',y)|) f(x',y) dx' \Big|^2 \\
&\leq C \left| \int \langle (x',y)\rangle^{\beta} \langle x-x'\rangle^{\beta} \hat{\Psi}(x'-x)(|(x,y)|-|(x',y)|) f(x',y) dx' \right|^2 \\
&\leq C \left(\int \langle (x',y)\rangle^{2\beta} |\hat{\Psi}(x-x')(|(x,y)|-|(x',y)|)| |f(x',y)|^2 dx' \right) \left(\int \langle x-x'\rangle^{2\beta} |\hat{\Psi}(x'-x)(|(x,y)|-|(x',y)|)| dx' \right).
\end{aligned}$$

Since $\int \langle x-x'\rangle^{2\beta} |\hat{\Psi}(x'-x)(|(x,y)|-|(x',y)|)| dx' < C$, where we allow the constant C to change from line to line, we have

$$\begin{aligned}
&\int \langle z\rangle^{2\beta} \Big| \int \hat{\Psi}(x'-x)(|(x,y)|-|(x',y)|) f(x',y) dx' \Big|^2 dz \\
&\leq C \int \int \langle (x',y)\rangle^{2\beta} |\hat{\Psi}(x'-x)(|(x,y)|-|(x',y)|)| |f(x',y)|^2 dx' dz \\
&\leq C \int \int \langle (x',y)\rangle^{2\beta} \left(\int (|\hat{\Psi}(x'-x)(|(x,y)|-|(x',y)|)|) dx \right) |f(x',y)|^2 dx' dy \\
&\leq C \int \int \langle (x',y)\rangle^{2\beta} |f(x',y)|^2 dx' dy
\end{aligned}$$

where for the last inequality we used that $|\int (|\hat{\Psi}(x'-x)(|(x,y)|-|(x',y)|)|) dx| < C$. A similar argument can be used when $\beta < 0$, using instead in the first step that for $\beta < 0$, $\langle z\rangle^{\beta} \leq C \langle x-x'\rangle^{-\beta} \langle (x',y)\rangle^{\beta}$.

A similar argument works for a derivative in the $z/|z|$ direction, and the argument can be iterated to get the lemma. $\square$

Lemma 6.9. *Let $\phi, \phi_1 \in C_c^\infty(\mathbb{R})$, with $(1-\phi)(1-\phi_1) = 1-\phi_1$ and $\phi_1(y) = 1$ if $|y| \leq y_M + 1$. If $D_z^\alpha (1-\phi(y)) g \in \langle z\rangle^{-\beta} L^2(\mathbb{R}^n)$ for $|\alpha| \leq l < \beta - 1/2$, $\phi(y) g \in \langle z\rangle^{-\beta} L^2(\mathbb{R}^n)$, $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$, and $\Psi \in C_b^\infty(\mathbb{R}^{n-1})$ with $\operatorname{supp}\Psi \cap \{\xi : |\xi| = |\lambda|/c_\pm\} = \emptyset$, then*

$$P\chi(z/|z|)(1-\phi_1(y))\Psi(D_x)(\Delta - (\lambda - i0)^2 c_0^{-2})^{-1} g \in \langle z\rangle^{1/2+\epsilon} L^2(\mathbb{R}^n)$$

for every $\epsilon > 0$ and $P \in \operatorname{Diff}_r^l$, with $l < \beta - 1/2$.

Proof. We give the proof for χ supported in $y/|z| > 0$; the proof for χ with support in $y/|z| < 0$ is similar.

We use a cut off-function, $\Psi_1 \in C_c^\infty(\mathbb{R})$ with $\Psi_1(|\xi|) \equiv 1$ when $|\xi| \leq \lambda/c_+$ and supported in a small neighborhood of that region, so that $\operatorname{supp}(\Psi_1(|\xi|)\Psi(\xi)) \subset \{|\xi| < \lambda/c_+\}$. We write

$$\begin{aligned}
&(1-\phi_1(y))\chi(z/|z|)\Psi(D_x)(\Delta - (\lambda - i0)^2 c_0^{-2})^{-1} g \\
&\qquad = (2\pi)^{1-n}(1-\phi_1(y))\chi(z/|z|) \int e^{ix\cdot\xi} \Psi(\xi)(\Psi_1(|\xi|) + 1 - \Psi_1(|\xi|)) \\
&\qquad\qquad (D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1} \hat{g}(\xi, \cdot)(y) d\xi. \qquad (39)
\end{aligned}$$

The main contribution is

$$(2\pi)^{1-n}(1-\phi_1(y))\chi(z/|z|) \int e^{ix\cdot\xi} \Psi(\xi)\Psi_1(|\xi|) e^{ix\cdot\xi} (D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1} \hat{g}(\xi, \cdot)(y) d\xi. \qquad (40)$$

Here we may write, for $y > y_M$ and $|\xi| < \lambda/c_+$,

$$(D_y^2 + |\xi|^2 - c_0^{-2}(\lambda - i0)^2)^{-1} \hat{g}(\xi, \cdot)(y) = e^{-iy\sqrt{\lambda^2/c_+^2 - |\xi|^2}} \tilde{g}(\xi) + g_1(\xi, y) \qquad (41)$$

where $D_y^k g_1 \in L^2(\mathbb{R}^{n-1}_\xi; \langle y\rangle^{-\beta+1}L^2(\mathbb{R}_y))$ for all k. Putting the first term of (41) into (40), we obtain

$$(2\pi)^{1-n}(1-\phi_1(y))\chi(z/|z|)\int \Psi(\xi)\Psi_1(|\xi|)e^{ix\cdot\xi}e^{-iy\sqrt{\lambda^2/c_+^2-|\xi|^2}}\tilde{g}(\xi)d\xi.$$

Note that if we apply $\frac{\partial}{\partial|z|}+i\lambda/c_+$, then the integrand vanishes on the critical set of the phase function. Since on the support of $\Psi\Psi_1$, $\tilde{g}\in H^l$, l an integer with $l<\beta-1/2$, we can integrate by parts to see that we have an element of $\langle z\rangle^{-1/2+\epsilon}L^2$.

For the tangential derivatives (in the $z/|z|$ directions), notice that if we have a derivative in a direction orthogonal to y, it commutes with $((D_y^2+|\xi|^2)-c_0^{-2}(\lambda-i0)^2)^{-1}$. That is, for example,

$$\begin{aligned}(-x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2})\int e^{ix\cdot\xi}\left(D_y^2+|\xi|^2-(\lambda-i0)^2c_0^{-2})^{-1}\hat{g}(\xi,\cdot)\right)(y)d\xi \\ =\int e^{ix\cdot\xi}\left(D_y^2+|\xi|^2-(\lambda-i0)^2c_0^{-2})^{-1}(\xi_1\frac{\partial}{\partial\xi_2}-\xi_2\frac{\partial}{\partial\xi_1})\hat{g}(\xi,\cdot)\right)(y)d\xi.\end{aligned}$$

By the decay properties of g and the regularity and decay properties of $(1-\phi)g$, this gives an element of $\langle z\rangle^{1/2+\epsilon}L^2(\mathbb{R}^n)$ after multiplication by $\chi(1-\phi_1)$, if $\beta>3/2$.

After applying a derivative of the form $y\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial y}$, as in the radial case the integrand vanishes on the critical set of the phase function, and so we can integrate by parts.

This argument can be iterated up to $l<\beta-1/2$.

The second term of (41) gives an element of $\langle z\rangle^{-\beta+l+1}L^2(\mathbb{R}^n)$, where l is the order of the derivative P.

On the support of $(1-\Psi_1(|\xi|))$, $((D_y^2+|\xi|^2)-(\lambda-i0)^2c_0^{-2})^{-1}$ is a smooth function of $|\xi|$, except near a finite number of points for which λ^2 is an eigenvalue of $c_0^2(D_y^2+|\xi|^2)$. Since the eigenfunctions of this operator are exponentially decreasing in y and χ is supported in $y/|z|>\delta>0$ for some $\delta>0$, these eigenfunctions do not contribute to the asymptotics here. Projecting off the eigenfunctions, we have

$$(1-\Psi_1)(D_y^2+|\xi|^2-(\lambda-i0)^2c_0^{-2})^{-1}\Pi_e:\langle y\rangle^{-\beta}L^2(\mathbb{R}_y)\to\langle y\rangle^{-\beta}L^2(\mathbb{R}_y)$$

with bound $C(|\xi|^2-C)^{-2}$. Therefore,

$$P(1-\phi_1(y))\chi(z/|z|)\int\Psi(\xi)(1-\Psi_1(|\xi|))(D_y^2+|\xi|^2-(\lambda-i0)^2c_0^{-2})^{-1}\hat{g}(\xi,\cdot)(y)d\xi\in\langle z\rangle^{-\beta+l}L^2(\mathbb{R}^n)$$

where we used the fact that the inverse Fourier transform is an isomorphism on L^2, and the regularity properties of $D_z^\alpha(1-\phi(y))g$. □

Lemma 6.10. *Let $f\in\langle z\rangle^{-\infty}L^2(\mathbb{R}^n)$, $(1-\phi(y))f\in\mathcal{S}(\mathbb{R}^n)$ for some $\phi\in C_c^\infty(\mathbb{R})$, and let $u=(\Delta-(\lambda-i0)^2)c^{-2})f$. If $\Psi_0\in C_b^\infty(\mathbb{R})$ is 0 in a neighborhood of $|\xi|=\lambda/c_\pm$ and $\chi_0\in C_c^\infty(\mathbb{S}_c)$, then there is a $w_0=\sum_\pm\tilde{\chi}_\pm\sum_{j=0}^{J-3}|z|^{-j-J+1}P_j\Psi_0(D_x)u$, with $P_j\in\mathrm{Diff}_r^j(\mathbb{R}^n)$, such that*

$$(\Delta-\lambda^2/c_0^2)w_0=\chi_0(z/|z|)V\Psi_0(D_x)u+e_0$$

where $e_0\in\langle z\rangle^{-2J+5/2+\epsilon}L^2(\mathbb{R}^n)$.

Proof. Let $\chi_0=\chi_++\chi_-$, with $\chi_\pm$ supported in $\pm y>0$. We will outline the proof for $\chi_0=\chi_+$, as the proof for $\chi_0=\chi_-$ is similar, and the functions can be added to get the general case.

Recall that on the support of χ_+, $V\sim\sum_{j\geq J}|z|^{-j}v_j(\frac{z}{|z|})$. We find $w=\sum_{j=0}^{J-3}w_{0j}$.

Let

$$w_{00}=(1-\phi_1(y))|z|^{-J+1}v_J\frac{ic_+}{\lambda(2J-2)}\chi_+\Psi_0(D_x)u$$

with $\phi_1\in C_c^\infty(\mathbb{R})$, $\phi_1\phi=\phi$, and $\phi_1(y)=1$ for $|y|\leq y_M$. Then, by Lemma 6.6

$$(\Delta-\lambda^2/c_0^2)w_{00}=|z|^{-J}v_J(\frac{z}{|z|})\chi_+\Psi_0(D_x)u+|z|^{-J-1}P_1'\Psi_0(D_x)u+|z|^{-J+1}\Psi_0(D_x)(Vu+f)+e_t$$

where $P_1' \in \mathrm{Diff}^1_r(\mathbb{R}^n)$ and $e_t \in \langle z\rangle^{-\infty}L^2(\mathbb{R}^n)$, so that $(\Delta - \lambda^2/c_0^2)w_{00} = \chi_+ V\Psi(D_x)u + e_{00}$, $e_{00} \in \langle z\rangle^{-J-1/2+\epsilon}L^2(\mathbb{R}^n)$.

In the same manner, we choose w_{01} so that

$$(\Delta - \lambda^2/c_0^2)w_{01} = |z|^{-J-1}v_{J+1}(\frac{z}{|z|})\chi_+\Psi_0(D_x)u - |z|^{-J-1}P_1'\Psi_0(D_x)u + |z|^{-J-2}P_2'\Psi_0(D_x)u + e_{01r}$$

with $P_2' \in \mathrm{Diff}^2_r(\mathbb{R}^n)$ and $e_{01r} \in \langle z\rangle^{-2J+1/2+\epsilon}L^2(\mathbb{R}^n)$. Then $(\Delta - \lambda^2/c_0^2)(w_{00}+w_{01}) = V\chi_+\Psi(D_x) + e_{01}$, $e_{01} \in \langle z\rangle^{-J-3/2+\epsilon}L^2(\mathbb{R}^n)$. This can be continued, with w_{0j} removing the terms in $\langle z\rangle^{-J+1/2+\epsilon-j}L^2(\mathbb{R}^n)$, modulo terms in $\langle z\rangle^{-J-1/2+\epsilon-j}L^2(\mathbb{R}^n)$, up to $j = J-3$. $\square$

We shall also need the following lemma.

Lemma 6.11. *If* $w(x,y) \in \langle z\rangle^{1/2+\epsilon}L^2(\mathbb{R}^n)$, $\Psi \in C_b^\infty(\mathbb{R}^{n-1})$, *and* $\mathrm{supp}\,\hat{w}(\xi,y) \cap \mathrm{supp}\,\Psi = \emptyset$, *then* $\Psi\widehat{Vw}(\xi,y) \in H^\infty(\mathbb{R}^{n-1}_\xi, \langle y\rangle^{-\infty}L^2(\mathbb{R}_y))$.

Proof. Observe that

$$\Psi(\xi)\widehat{Vw}(\xi,y) = \Psi(\xi)\int\int e^{-ix\cdot\xi}V(x,y)e^{ix\cdot\eta}\hat{w}(\eta,y)d\eta dx \tag{42}$$

$$= \sum_j \Psi(\xi)\int\int \Psi_j(\xi,\eta)(\xi_j-\eta_j)^{-1}e^{-ix\cdot(\xi-\eta)}D_{x_j}V(x,y)\hat{w}(\eta,y)d\eta dx \tag{43}$$

where Ψ_j is a partition of unity with $\xi_j \neq \eta_j$ on $\mathrm{supp}\,\Psi_j$. We may repeat this integration by parts as many times as desired. Since $|D_x^\alpha V(x,y)| \leq C_\alpha\langle z\rangle^{-J-|\alpha|}$, the lemma follows. $\square$

Proposition 6.3. *Suppose* $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$, $\Psi \in C_c^\infty(\mathbb{R}^{n-1})$ *has* $\mathrm{supp}\,\Psi \cap \{\xi : |\xi| = |\lambda|/c_\pm\} = \emptyset$, $\mathrm{supp}\,\Psi \cap \{\xi : |\xi| = \kappa_j(\lambda)\} = \emptyset$, $j = 1,2,...,T(\lambda)$, *and if* $(x,y)/|(x,y)| \in$ *supp*$(1-\chi)$, $y > 0$, *then* $\pm\lambda x/(c_+|z|) \notin \mathrm{supp}\,\Psi$. *Moreover, suppose if* $(x,y)/|(x,y)| \in$ *supp*$(1-\chi)$, $y < 0$, *then* $\pm\lambda x/(c_-|z|) \notin \mathrm{supp}\,\Psi$, *and* $f \in \langle z\rangle^{-\infty}L^2(\mathbb{R}^{n-1})$, $(1-\phi(y))f \in \mathcal{S}(\mathbb{R}^n)$ *for some* $\phi \in C_c^\infty(\mathbb{R})$. *Then*

$$\chi(\frac{z}{|z|})\Psi(D_x)(\Delta - (\lambda - i0)^2/c_0^2)^{-1}f = e^{-i\lambda|z|/c_0}\chi(\frac{z}{|z|})|z|^{-(n-1)/2}(a_0(\frac{z}{|z|}) + \mathcal{O}(|z|^{-1}))$$

with $\chi a_0 \in C_c^\infty(\mathbb{S}_c^{n-1})$.

Proof. Recall that if $u = (\Delta - (\lambda - i0)^2/c^2)^{-1}f$, then $u = (\Delta - (\lambda-i0)^2/c_0^2)^{-1}(Vu+f)$. If $\Psi(\xi)\widehat{Vu+f}(\xi,y)$ were in $C^\infty(\mathbb{R}^{n-1};\langle y\rangle^{-\infty}L^2(\mathbb{R}))$, then using

$$(\Delta - (\lambda-i0)^2c_0^{-2})^{-1}g = (2\pi)^{1-n}\int e^{ix\cdot\xi}\left((|\xi|^2 + D_y^2 - (\lambda-i0)^2c_0^{-2})^{-1}\hat{g}(\xi,\cdot)\right)(y)d\xi \tag{44}$$

and stationary phase, we would be done. However, it is not clear that $\widehat{Vu}$ should be in $C^\infty(\mathbb{R}^{n-1};\langle y\rangle^{-\infty}L^2(\mathbb{R}))$. We shall show that $\chi\Psi(D_x)u$ can be written as a sum of two terms: one vanishing faster than u at infinity, and another of the form $(\Delta - (\lambda-i0)^2/c_0^2)^{-1}g_k$, where $\Psi(\xi)\hat{g}_k(\xi,y) \in C^k(\mathbb{R}^{n-1};\langle y\rangle^{-k}L^2(\mathbb{R}))$, where we can make k as large as desired. Then (44) and stationary phase will finish the proof.

To do this, we follow an iterative procedure. The first step has been done in Lemma 6.10. We will iteratively construct functions w_l which have the property that $(\Delta - \lambda^2/c_0^2)(u - w_l)$ improves with increasing l in an appropriate sense.

Let $\Psi_0, \Psi_1, \Psi_2, ... \in C_c^\infty(\mathbb{R}^{n-1})$ be such that, for all i, $\Psi_i \equiv 1$ on the support of Ψ, $\Psi_{i+1}\Psi_i = \Psi_{i+1}$, and Ψ_i satisfies the support requirements placed on Ψ in the statement of the Proposition. Let $\chi_0, \chi_1, \chi_2, ... \in C_c^\infty(\mathbb{S}_c^{n-1})$ be such that $\chi_0\chi = \chi$ and $\chi_{i+1}\chi_i = \chi_i$, $i = 0,1,2....$ Let w_0 be the function constructed in Lemma 6.10 for this Ψ_0 and χ_0. Using the notation of that lemma, let

$$t_0 = (1-\chi_0)V\Psi_0(D_x)u + f - e_0 + V(1-\Psi_0(D_x))u.$$

Note that the first three terms are in $\langle z\rangle^{-2J+5/2+\epsilon}L^2(\mathbb{R}^n)$, where we use the support properties of χ_0 and Ψ_0 along with an integration by parts argument as in the proof of Lemma 6.4 to obtain the result for the first term. Additionally, the support of the Fourier transform in the x variables of $(1-\Psi_0(D_x))u$ is disjoint from the support of Ψ_1. Then, using Lemmas 6.9 and 6.11, we have $P\chi_1\Psi_1(D_x)(\Delta-(\lambda-i0)^2/c_0^2)^{-1}t_0 \in \langle z\rangle^{1/2+\epsilon}L^2(\mathbb{R}^n)$, for all $P\in \mathrm{Diff}_r^l(\mathbb{R}^n)$, $l\le 2J-4$. Since $u=w_0+(\Delta-(\lambda-i0)^2/c_0^2)^{-1}t_0$, which can be seen by a modification of the proof of Proposition 6.1 and the uniqueness result (Proposition 4.2), this in turn means that $P\Psi_1(D_x)w_0\in\langle z\rangle^{-J+3/2+\epsilon}L^2(\mathbb{R}^n)$ for all $P\in\mathrm{Diff}_r^l$, $l\le 2J-4$, using Lemma 6.8.

Given w_0, t_0 as above, we now iteratively construct w_l for $l\ge 1$ such that

$$(\Delta-\lambda^2/c_0^2)w_l = V\chi_l\Psi_l(D_x)u+e_l = V\chi_l\Psi_l(D_x)w_{l-1}+V\chi_l\Psi_l(D_x)(\Delta-(\lambda-i0)^2/c_0^2)^{-1}t_{l-1}+e_l,$$

where $e_l\in\langle z\rangle^{-J+1/2+\epsilon-l(J-2)}L^2(\mathbb{R}^n)$ and t_l is defined by

$$t_l = V\left((1-\chi_l)\Psi_l(D_x)+(1-\Psi_l(D_x))\right)w_{l-1} + V\left((1-\Psi_l(D_x))+(1-\chi_l)\Psi_l(D_x)\right)(\Delta-(\lambda-i0)^2/c_0^2)^{-1}t_{l-1}-e_l+f.$$

Then

$$u = w_l+(\Delta-(\lambda-i0)^2/c_0^2)^{-1}t_l.$$

Moreover, $t_l=t_l'+t_l''$, where $\Psi_{l+1}(\xi)\hat{t_l'}(\xi,y)\in C^\infty(\mathbb{R}^{n-1}_\xi,\langle y\rangle^{-\infty}L^2(\mathbb{R}))$, $t_l''\in\langle z\rangle^{-(l+1)(J-2)-J+1/2+\epsilon}L^2(\mathbb{R}^n)$, and $P\Psi_{l+1}(D_x)w_l\in\langle z\rangle^{-J+3/2+\epsilon}L^2(\mathbb{R}^n)$ for all $P\in\mathrm{Diff}_r^m(\mathbb{R}^n)$, $m\le(l+2)(J-2)$. Additionally, $\operatorname{supp} w_l(z)\subset \operatorname{supp}\chi_l(z/|z|)$.

Supposing that w_{l-1}, t_{l-1} are as above, we show how to construct w_l. Since

$$P\chi_l\Psi_l(D_x)w_{l-1}\in\langle z\rangle^{-J+3/2+\epsilon}L^2(\mathbb{R}^n) \text{ and } P\chi_l\Psi_l(D_x)(\Delta-(\lambda-i0)^2/c_0)^{-1}t_{l-1}\in\langle z\rangle^{1/2+\epsilon}L^2(\mathbb{R}^n)$$

for all $P\in\mathrm{Diff}_r^{(l+1)(J-2)}$, we can, just as in Lemma 6.10, find $w_l=\sum_{j=0}^{(l+1)(J-2)-1}w_{lj}$ so that

$$(\Delta-\lambda^2/c_0^2)w_l = V\chi_l\Psi_l(D_x)w_{l-1}+V\chi_l\Psi_l(D_x)(\Delta-(\lambda-i0)^2/c_0^2)^{-1}t_{l-1}+e_l$$

with $e_l\in\langle z\rangle^{-J+1/2+\epsilon-(l+1)(J-2)}L^2(\mathbb{R}^n)$. Let $t_l'=V(1-\Psi_l(D_x))(w_l+(\Delta-(\lambda-i0)^2)^{-1}t_{l-1}))$; then $\Psi_{l+1}(\xi)\hat{t_l'}(\xi,y)\in C^\infty(\mathbb{R}^{n-1}_\xi,\langle y\rangle^{-\infty}L^2(\mathbb{R}))$ by Lemma 6.11. Since $\operatorname{supp} w_{l-1}(z)\subset\operatorname{supp}\chi_{l-1}(z/|z|)$, by Lemma 6.7, $V(1-\chi_l)\Psi_l(D_x)w_{l-1}\in\langle z\rangle^{-\infty}L^2(\mathbb{R}^n)$, and

$$V(1-\chi_l)\Psi_l(D_x)(\Delta-(\lambda-i0)^2c_0^{-2})^{-1}t_{l-1}\in\langle z\rangle^{-(l+1)(J-2)-1/2-J+\epsilon}L^2(\mathbb{R}^n)$$

using the support properties of Ψ_l and χ_l and an integration by parts argument as in Lemma 6.9. Thus $t_l''=t_l-t_l'\in\langle z\rangle^{-(l+1)(J-2)-J+1/2+\epsilon}L^2(\mathbb{R}^n)$.

Note that

$$w_{lj}=|z|^{-j-J+1}P_{lj}(1-\phi(y))\Psi_l(D_x)[w_{l-1}+(\Delta-(\lambda-i0)^2/c_0^2)^{-1}t_{l-1}] = |z|^{-j-J+1}P_{lj}(1-\phi(y))\Psi_l(D_x)u, \tag{45}$$

with $P_{lj}\in\mathrm{Diff}_r^j(\mathbb{R}^n)$, $\phi\in C_c^\infty(\mathbb{R})$. Since $(\Delta-\lambda^2/c_0^2)(u-w_l)=t_l$, we have

$$u=w_l+(\Delta-(\lambda-i0)^2/c_0^2)^{-1}t_l. \tag{46}$$

Note that $P\Psi_{l+1}(D_x)(\Delta-(\lambda-i0)^2/c_0^2)^{-1}t_l\in\langle z\rangle^{1/2+\epsilon}L^2(\mathbb{R}^n)$ for $P\in\mathrm{Diff}_r^{(l+2)(J-2)}(\mathbb{R}^n)$. Using (45), (46), and Lemma 6.8, this in turn means that $P\Psi_{l+1}(D_x)w_l\in\langle z\rangle^{3/2+\epsilon-J}L^2(\mathbb{R}^n)$ for all $P\in\mathrm{Diff}_r^{(l+2)(J-2)}(\mathbb{R}^n)$. Thus, for any $l\ge 1$, w_l and t_l can be constructed to have the desired properties.

To prove the proposition, we use (46). Since $t_l=t_l'+t_l''$ with $\Psi_{l+1}(\xi)\hat{t_l'}(\xi,y)\in C^\infty(\mathbb{R}^{n-1}_\xi,\langle y\rangle^{-\infty}L^2(\mathbb{R}))$, $t_l''\in\langle z\rangle^{-(l+1)(J-2)-J+1/2+\epsilon}L^2(\mathbb{R}^n)$, we have $\Psi(\xi)\hat{t}_l(\xi,y)\in H^s(\mathbb{R}^{n-1};\langle y\rangle^{-(l+1)(J-2)+1/2+\epsilon-J+s}L^2(\mathbb{R}))$ for $s<(l+1)(J-2)+J-1/2-\epsilon$. Then, using equation (44) and stationary phase, we see that

$$\chi\Psi(D_x)(\Delta-(\lambda-i0)^2/c_0^2)^{-1}t_l = \chi e^{-i\lambda|z|/c}|z|^{-(n-1)/2}(a_0(\frac{z}{|z|})+\mathcal{O}(|z|^{-1}))$$

with $a_0 \in C^{(l+1)(J-2)+J-n-4}(\mathbb{S}_c)$ when l is sufficiently large.

To finish, then, we only need show that $\Psi(D_x)w_l$ is of order $|z|^{-(n+1)/2}$. But we recall that

$$D^{\alpha}_{z/|z|}w_l,\ |z|(\frac{\partial}{\partial |z|} + i\lambda/c)D^{\alpha}_{z/|z|}w_l \in \langle z\rangle^{-J+3/2+\epsilon}L^2(\mathbb{R}^n).$$

This in turn means that $\Psi(D_x)w_l$ has the same properties, and that $\Psi(D_x)w_l = \mathcal{O}(|z|^{-(n-1)/2-J+1+\epsilon})$. ☐

6.4. **The structure of the scattering matrix.** Combining the construction of the approximate Poisson operator of Section 5 and Theorem 1.3, we have now proved Theorem 1.1. For completeness, we remark that the construction of the approximation of $P(\lambda)\Pi_j$, $1 \le j \le T(\lambda)$ in Section 5.2, Corollary 6.1, and Theorem 1.3 prove

Proposition 6.4. *Let c, c_0 satisfy the general conditions of Section 2 and either hypothesis (H1) or (H2). Let $A(\lambda) = (A_{ij}(\lambda))$, $0 \le i, j \le T(\lambda)$. Then, for $j > 0$, $A_{jj}(\lambda)$ is a Fourier integral operator associated with the antipodal mapping on $\mathbb{S}^{n-2}$, and $A_{ij}(\lambda)$ is a smoothing operator when $i \ne j$.*

7. The Inverse Problem

We recall our central inverse result, Theorem 1.2:

Theorem . *Suppose c and c_0 satisfy the general assumptions of Section 2, as well as either hypothesis (H1) or (H2), and $n \ge 3$. Then, if $c_+ = c_-$, the asymptotic expansion at infinity of $c - c_0$ is uniquely determined by c_0 and the transmitted singularities of the main part of the scattering matrix at fixed nonzero energy. If $c_+ < c_-$, then the asymptotic expansion is uniquely determined by c_0 and the reflected singularities of the main part of the scattering matrix at fixed nonzero energy.*

In proving the results for the inverse problem, we use the techniques of [18, 19] and much of their language. We recall the arguments from these papers, noting the adjustments that must be made for the stratified case.

Theorem 1.2 follows from the following theorem, which is somewhat stronger.

Theorem 7.1. *Suppose $n \ge 3$, c_1 and c_2 satisfy the general requirements of Section 2, and either (H1) or (H2), for the same c_0. Let $S_1(\lambda)$, $S_2(\lambda)$ be the corresponding scattering matrices for some $\lambda \in \mathbb{R}\setminus\{0\}$. If, for $c_+ = c_-$, the transmitted part of the main part of $S_1(\lambda) - S_2(\lambda)$ is of order $-l$, then $c_1(z) - c_2(z) = \mathcal{O}(|z|^{-l-1})$. If for $c_+ < c_-$ the reflected part of $S_1(\lambda) - S_2(\lambda)$ is of order $-l$, then $c_1(z) - c_2(z) = \mathcal{O}(|z|^{-l-1})$.*

Proof. Let $S^k_{\mathrm{cl},s}(\mathbb{R}^n)$ be the set of functions which, for $|z| > 1$, are of the form $\sum_{j\ge 0}|z|^{k-j}a_j(z/|z|)$, where $a_j(z/|z|) \in C^\infty_b(\mathbb{S}^{n-1}\setminus\{(\overline{\omega},0)\})$.

Suppose that $c_1 - c_2 = W \in S^{-k}_{\mathrm{cl},s}(\mathbb{R}^n)$ and that the scattering matrices associated to $c_1^2\Delta$ and $c_2^2\Delta$ have the same transmitted (if $c_+ = c_-$) or reflected (if $c_+ \ne c_-$) singular parts, to order $l \ge k$. Then we shall show that actually $c_1 - c_2 \in S^{-k-1}_{\mathrm{cl},s}(\mathbb{R}^n)$, and thus by induction $c_1 - c_2 \in S^{-l-1}_{\mathrm{cl},s}(\mathbb{R}^n)$.

If $c_1 - c_2 \in S^{-k}_{\mathrm{cl},s}(\mathbb{R}^n)$, then $\lambda^2(c_1^{-2} - c_2^{-2}) = \lambda^2 c_1^{-2}c_2^{-2}(c_1 + c_2)(c_2 - c_1) = U$, with

$$U_{|y>y_M} \sim \sum_{j\ge k}|z|^{-j}U_{-j,+}(z/|z|),\ \ U_{|y<-y_M} \sim \sum_{j\ge k}|z|^{-j}U_{-j,-}(z/|z|),$$

and $U = \mathcal{O}(|z|^{-k})$, where $U_{-j,\pm} \in C^\infty_b(\mathbb{S}^{n-1}_\pm)$. Let

$$W_{-j}(\overline{\omega}, \omega_n) = \begin{cases} U_{-j,+}(\overline{\omega}, \omega_n) & \text{if } \omega_n > 0 \\ U_{-j,-}(\overline{\omega}, \omega_n) & \text{if } \omega_n < 0 \end{cases}$$

Note then that the first $k-2$ terms in the construction of the Poisson operator carried out in Section 5 are the same, and the difference comes in the $k-1$st term.

Although many of the underlying techniques are the same, we shall treat the cases $c_+ = c_-$ and $c_+ \ne c_-$ serially.

We begin with the case $c_+ = c_-$. In the construction of the Poisson operator, the transmitted parts (that is, in the lower hemisphere) of the $k-1$st terms differ by

$$\frac{i|z|^{1-k}}{2\lambda c_+ (\sin s)^{k-1}} T_+(\omega_n, \lambda) \int_0^s W_{-k}(s', \theta; \omega)(\sin s')^{k-2} ds', \tag{47}$$

almost as in (2.3) of [18]. Here $T_+(\lambda, \omega_n)$ is the transmission coefficient determined by equations (14)-(16). We remark that in case $c_+ = c_-$, $T_+{=}T_-$. The transmission coefficient must be nonzero for $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $\omega_n \neq 0$, $0 < \omega_n < 1$. Therefore, as described in [18, Section 4], we can recover from the difference of the transmitted parts of the scattering matrices

$$\int_0^\pi W_{-k}(s, \theta; \omega)(\sin s)^{k-2} ds$$

as long as $\omega = (\overline{\omega}, \omega_n)$ with $\omega_n \neq 0$.

If the transmitted parts of the two scattering matrices are the same to order $k-1$, then

$$I_k = \int_0^\pi W_{-k}(s, \theta; \omega)(\sin s)^{k-2} ds = 0. \tag{48}$$

Since this is true for all ω with $\omega_n \neq 0$, we can differentiate with respect to the starting point twice, use $\sin^2 s + \cos^2 s = 1$, and find that $I_{k-2} = \int_0^\pi W_{-k}(s, \theta; \omega)(\sin s)^{k-4} ds = 0$ as well (see [19, Section 5]). Therefore, if k is even, we reduce eventually to the case with $k-2=0$ and if k is odd, to $k-2=1$. When k is even, differentiating one more time with respect to the starting point shows that W_{-k} is even; for odd k two more differentiations show that W_{-k} is odd.

When k is even, we obtain

$$\int_\gamma W_{-k} = 0$$

for each closed geodesic γ by joining together two half-geodesics. However, by [15, Theorem 4.7], for $n \geq 3$ the x-ray transform on $\mathbb{S}^{n-1}$ with domain restricted to smooth even functions is 1-1. Although W_{-k} may have a jump discontinuity at $\omega_n = 0$, it is smooth elsewhere. As in the proof of [15, Corollary 4.19], by first taking a convolution with W_{-k} we may assume that W_{-k} is smooth and, applying the theorem, it is thus 0.

If k is odd, we consider $\frac{z_n}{|z|} W_{-k}$, which is even. Since for each geodesic beginning at $z_n = 0$, $\sin s$ is a constant multiple of $z_n/|z|$, we obtain $\int_\gamma \frac{z_n}{|z|} W_{-k} = 0$ for each geodesic γ, and again $W_{-k} = 0$.

When $c_+ \neq c_-$, we use the reflected singularities in the inverse problem. Recalling that $W \in S^{-k}_{\mathrm{cl},s}$, we can recover from the reflected singularities, when $\omega_n > 0$,

$$R_+(\omega_n, \lambda) \left(\int_0^{s_0} W_{-k}(s, \theta; \omega)(\sin s)^{k-2} ds + \int_{s_0}^\pi W_{-k}(s', \theta; \omega)(\sin s')^{k-2} ds'. \right) \tag{49}$$

The first integral is along a geodesic originating at ω and continuing to $\{(\overline{\phi}, 0)\} \subset \mathbb{S}^{n-1}$; the second integral is along the reflection of the first geodesic when it meets $\phi_n = 0$ and the path of integration ends at the point $(-\overline{\omega}, \omega_n)$. The variable s' is the distance from the point $(\overline{\omega}, -\omega_n)$.

It is, however, more convenient to think of the sum (49) as the single integral

$$R_+(\omega_n, \lambda) \int_0^\pi \tilde{W}_{-k+}(s, \theta; \omega)(\sin s)^{k-2} ds \tag{50}$$

where

$$\tilde{W}_{-k+}(\phi) = \begin{cases} W_{-k}(\phi) & \text{if } \phi_n \geq 0 \\ W_{-k}(\overline{\phi}, -\phi_n) & \text{if } \phi_n < 0 \end{cases} \tag{51}$$

and s is the distance from ω. It is fairly straightforward to see by symmetry that (49) and (50) are the same.

If we can show that (50) is sufficient for recovering $\int_0^\pi \tilde{W}_{-k,+}(s,\theta;\omega)(\sin s)^{k-2}ds$ for all ω with $\omega_n > 0$, then the analysis used in the case $c_+ = c_-$ will show that $\tilde{W}_{-k,+}$ is 0 if the reflection coefficients agree to order $-k$.

It suffices that $R_+(\omega_n, \lambda)$ is 0 for at most an isolated set of ω_n with $0 < \omega_n \leq 1$, for we can obtain the integrals for these isolated values of ω_n by continuity. We recall that for $0 < \omega_n < \sqrt{1 - c_+^2/c_-^2}$, $|R_+(\omega_n, \lambda)| = 1$ ([30, Chapter 3]). Moreover, because $c_0(y) - c_\pm$ is compactly supported for $\pm y > 0$, for fixed $\lambda \in \mathbb{R}$, $R_+(\omega_n, \lambda)$ can be extended to a meromorphic function of ω_n in a neighborhood of $0 < \omega_n < 1$, except near $\omega_n = \sqrt{1 - c_+^2/c_-^2}$, where it is a meromorphic function of $(1 - c_-^2/c_+^2 + c_-^2\omega_n^2/c_+^2)^{1/2}$. Therefore, its zeros are isolated, and we have shown that it is possible to recover $W_{-k}(\omega)$ for $\omega_n > 0$.

A very similar analysis works for the lower hemisphere, proving the theorem. □

We remark that this proof shows that if $c_1 - c_2 \in S_{\mathrm{cl}}(\mathbb{S}^{n-1})$, then the main part of $S_1(\lambda) - S_2(\lambda)$ is of order $-k+1$.

Corollary 7.1. *Let c, c_0 satisfy the general conditions of Section 2, and either (H1) or (H2), and let $n \geq 3$. Then c_+, c_-, and the main part of the scattering matrix at nonzero fixed energy determine c modulo terms vanishing faster than the reciprocal of any polynomial at infinity.*

Proof. We need only show that c_0 can be recovered from c_+, c_-, and knowledge of the scattering matrix at fixed energy. The leading order singularity of the scattering matrix $A(\lambda)$ determines and is determined by $R_\pm(\lambda, \omega_n)$, $T_\pm(\lambda, \omega_n)$, λ and $c_\pm$, where R_+, T_+ are defined by equations (14-16), and a similar definition gives R_-, T_-, just as in one-dimensional scattering theory (see e.g. [8]).

Fix $\lambda \in \mathbb{R} \setminus \{0\}$. We can think of equation (14) in the slightly more general form

$$(D_y^2 - \lambda^2(1/c_0^2 - 1/c_+^2) - k^2)\phi = 0, \tag{52}$$

a Schrödinger operator with potential $-\lambda^2(1/c_0^2 - 1/c_+^2)$ which is either compactly supported (if $c_+ = c_-$) or "steplike" (if $c_+ < c_-$). We can define the reflection and transmission coefficients $\tilde{R}_\pm(k)$, $\tilde{T}_\pm(k)$, for (52) as usual, as in (14-16), and $\tilde{R}_\pm(k) = R_\pm(\lambda, \omega_n)$, $\tilde{T}_\pm(k) = T_\pm(\lambda, \omega_n)$, when $k = \lambda\omega_n/c_+$. Moreover, $\tilde{R}_\pm$, $\tilde{T}_\pm$ are meromorphic functions of $k \in \mathbb{C}$ if $c_+ = c_-$, and if $c_+ < c_-$, they are meromorphic functions on $\hat{Z}$, the Riemann surface on which k and $(k^2 - \lambda^2/c_+^2 + \lambda^2/c_-^2)^{1/2}$ are single-valued holomorphic functions. Therefore, knowing $R_+(\lambda, \omega_n)$, $T_+(\lambda, \omega_n)$ for $0 < \omega_n < \sqrt{1/c_+^2 - 1/c_-^2}$ determines $\tilde{R}_+(k)$, $\tilde{T}_+(k)$ on the whole plane (if $c_+ = c_-$) or $\hat{Z}$ (if $c_+ < c_-$). This in turn determines the eigenvalues of $D_y^2 - \lambda^2(1/c_0^2 - 1/c_+^2)$ and the norming constants. These, together with $c_\pm$ and $\tilde{R}_+$, determine the potential $-\lambda^2(1/c_0^2 - 1/c_+^2)$ (e.g. [11, 8]). □

References

[1] I. Beltiţă, *Inverse scattering in a layered medium*, C.R. Acad. Sci Paris Sér. I Math **329** (1999), no. 10, 927-932.

[2] I. Beltiţă, *Inverse scattering in a layered medium*, preprint.

[3] M. Ben-Artzi, Y. Dermenjian, and J.-C. Guillot, *Acoustic waves in perturbed stratified fluids: a spectral theory*, Comm. Partial Differential Equations **14** (4) (1989), 479-517.

[4] A. Boutet de Monvel-Berthier and D. Manda, *Spectral and scattering theory for wave propagation in perturbed stratified media,* J. Math Anal. Appl. **191** (1995), 137-167.

[5] T. Christiansen, *Scattering theory for perturbed stratified media*. Journal d'Analyse Mathématique **76** (1998), 1-44.

[6] T. Christiansen and M.S. Joshi, *Higher order scattering on asymptotically Euclidean manifolds*. To appear, Canadian Journal of Mathematics.

[7] T. Christiansen and M.S. Joshi, *Recovering asymptotics at infinity of perturbations of stratified media*. Proceedings of Journées Équations aux Dérivées Partielles, Nantes, France, 2000.

[8] A. Cohen and T. Kappeler, *Scattering and inverse scattering for steplike potentials in the Schrödinger equation*, Indiana Univ. Math. J., **34** (1985), 127-180.

[9] S. DeBièvre and D.W. Pravica, *Spectral anslysis for optical fibres and stratified fluids I: the limiting absorption principle*, J. Functional Analysis **98** (1991), 404-436.

[10] S. DeBièvre and D.W. Pravica, *Spectral analysis for optical fibres and stratified fluids II: Absence of eigenvalues*, Comm. Partial Differential Equations **17** (1&2) (1992), 69-97.
[11] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Commun. Pure Appl. Math. **32** (1979), 121-251.
[12] Y. Dermenjian and J.-C. Guillot, *Théorie spectrale de la propagation des ondes acoustiques dans un milieu stratifié perturbé*, J. Differential Equations **62** (3) (1986), 357-409.
[13] C. Gérard, H. Isozaki, and E. Skibsted, *Commutator algebra and resolvent estimates*, volume 23 of *Advanced studies in pure mathematics,* p. 69-82, 1994.
[14] J.-C. Guillot and J. Ralston, *Inverse scattering at fixed energy for layered media*, J. Math. Pures Appl. (9) **78** (1999), 27-48.
[15] S. Helgason, *Groups and Geometric Analysis*, Academic Press, Orlando, 1984.
[16] H. Isozaki, *Inverse scattering for wave equations in stratified media*, Journal of Differential Equations **138** (1997), 19-54.
[17] M.S. Joshi, *Recovering asymptotics of Coulomb-like potentials from fixed energy scattering data,* S.I.A.M. J. Math. Anal. **30** (1999), No. 3, 516-526.
[18] M.S. Joshi, *Explicitly recovering asymptotics of short range potentials,* Communications on Partial Differential Equations **25**, (2000), Nos. 9 & 10, 1907-1923.
[19] M.S. Joshi and A. Sá Barreto, *Recovering asymptotics of short range potentials,* Comm. Math. Phys. **193** (1998), 197-208.
[20] M.S. Joshi and A. Sá Barreto, *Recovering asymptotics of metrics from fixed energy scattering data*, Invent. Math. **137** (1999) 127-143.
[21] M.S. Joshi and A. Sá Barreto, *Determining asymptotics of magnetic potentials from fixed energy scattering data,* Asymptotic Analysis **61** Vol. 21, Number 1, (1999) 61-70.
[22] R.B. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidean spaces*, in *Spectral and Scattering Theory* (M. Ikawa, ed), p. 85-130, Marcel Dekker, New York, 1994.
[23] R.B. Melrose and M. Zworski, *Scattering Metrics and Geodesic Flow at Infinity,* Invent. Math. **124** (1996), 389-436.
[24] A. Vasy, *Asymptotic behavior of generalized eigenfunctions in N-body scattering*, J. Funct. Anal. **148** (1997), no. 1, 170–184.
[25] A. Vasy, *Structure of the resolvent for three-body potentials*, Duke Math. J. **90** (1997), no. 2, 379–434.
[26] A. Vasy, *Propagation of singularities in Euclidean many-body scattering in the presence of bound states*, Journées Équations aux Dérives Partielles" (Saint-Jean-de-Monts, 1999), Exp. No. XVI, 20 pp., Univ. Nantes, Nantes, 1999.
[27] R. Weder, *The limiting absorption principle at thresholds*, J. Math. Pures et Appl. **67** (1988), 313-338.
[28] R. Weder, *Spectral and Scattering Theory for Wave Propagation in Perturbed Stratified Media*, Springer-Verlag, New York, 1991.
[29] R. Weder, *Multidimensional inverse problems in perturbed stratified media*, J. Differential Equations **152** (1999), no. 1, 191–239.
[30] C. Wilcox, *Sound Propagation in Stratified Fluids*, Applied Mathematical Sciences **50**. Springer-Verlag, New York, Berlin, Heidelberg.

Department of Mathematics, University of Missouri Columbia, Missouri 65211 U.S.A.
e-mail: tjc@math.missouri.edu

Royal Bank of Scotland Group Risk, Level 9, 135 Bishopsgate, London, EC2M 3UR, U.K.
e:mail: markjoshi@alum.mit.edu